Contents

2 Dynamics of One-Parameter Families

In this chapter, our goal is to analyze how the dynamics of a one-parameter family $f_\lambda(x)$ changes as we vary the parameter λ: in particular, we want to study the locations of fixed points and periodic cycles along with their behavior (attracting, repelling, or neutral). We begin by constructing bifurcation diagrams that we can use to better understand changes in the orbit structures in families of maps such as on the quadratic family $q_c(x) = x^2 + c$, and then analyze the most common types of fixed-point bifurcations that can occur.

We then study attracting cycles of one-parameter families, and establish a number of useful results involving the existence (or lack thereof) of attracting cycles. Finally, we will combine these ideas and use them to study the orbit diagram of a one-parameter family, which will motivate our future study of chaotic dynamical systems.

2.1 Bifurcations in One-Parameter Families

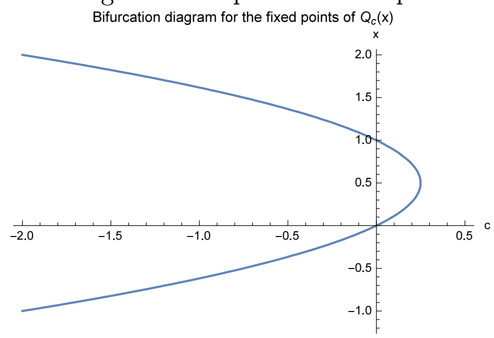
- Definition: A one-parameter family has the form $f_\lambda(x)$ where the function f depends smoothly on the parameter λ.
  - Examples: $f_\lambda(x) = \lambda e^x$, $g_\lambda(x) = x^3 + \lambda x + \lambda^2$, and $h_\lambda(x) = \lambda\sin(\lambda x)$ are all one-parameter families.
- Linear one-parameter families do not have especially interesting dynamics, so we will begin by studying the quadratic maps of the form $q_c(x) = x^2 + c$.
  - Although it may seem that this family is only a small subset of all the quadratic maps, which have the general form $f(x) = ax^2 + bx + c$, in fact as we will see later every quadratic map can be put in this form using a change of variables.
- We will see several examples of bifurcations: changes in the qualitative orbit structure of the function that occur at a particular "threshold value" of the parameter λ.
  - There are several different types of bifurcations, which we will then analyze more generally.

2.1.1 Motivation: The Quadratic Family $q_c(x) = x^2 + c$

- So, consider the quadratic map $q_c(x) = x^2 + c$.
- We start by analyzing the behavior of the fixed points:
  - We have $q_c(x) - x = x^2 - x + c$, which has roots $p_+ = \frac{1+\sqrt{1-4c}}{2}$ and $p_- = \frac{1-\sqrt{1-4c}}{2}$.
  - If $c > \frac{1}{4}$, then there are no real-valued fixed points.
  - If $c = \frac{1}{4}$, then the two points p_+ and p_- coincide at $\frac{1}{2}$, so there is only one fixed point. Since $q'_{1/2}(\frac{1}{2}) = 1$, this fixed point is neutral, and since $q''_{1/2}(\frac{1}{2}) = 2$, this neutral fixed point is weakly attracting on the left and weakly repelling on the right.
  - If $c < \frac{1}{4}$, then the two fixed points p_+ and p_- are distinct. We compute $q'_{1/2}(p_+) = 1 + \sqrt{1-4c} > 1$, so p_+ is always repelling. Additionally, $q'_{1/2}(p_-) = 1 - \sqrt{1-4c}$, so p_- is attracting for $-\frac{3}{4} < c < \frac{1}{4}$, neutral for $c = -\frac{3}{4}$, and repelling for $c < -\frac{3}{4}$.
  - When $c = -\frac{3}{4}$, we have $p_- = -\frac{1}{2}$, and since $q'_{-3/4}(-\frac{1}{2}) = -1$, to classify this neutral fixed point we need to look at $g(x) = q^2_{-3/4}(x) = x^4 - \frac{3}{2}x^2 - \frac{3}{16}$. As $g'(-\frac{1}{2}) = 1$, $g''(-\frac{1}{2}) = 0$, and $g'''(-\frac{1}{2}) = -12$, this tells us $-\frac{1}{2}$ is weakly attracting as a fixed point of g and hence of $q_{-3/4}$.
- We can also analyze the behavior of the 2-cycle:
  - We have $\frac{q_c(q_c(x)) - x}{q_c(x) - x} = \frac{x^4 + 2cx^2 - x + (c^2+c)}{x^2 - x + c} = x^2 + x + (c+1)$, whose roots are $r_+ = \frac{-1+\sqrt{3-4c}}{2}$ and $r_- = \frac{-1-\sqrt{3-4c}}{2}$.
  - If $c > -\frac{3}{4}$ then there is no real-valued 2-cycle, and if $c = -\frac{3}{4}$ then there is also no 2-cycle since the two points r_+ and r_- coincide (at the value $-\frac{1}{2}$): indeed, they also coincide with the fixed point p_- in this case.
  - We can also compute $q'_c(r_+) \cdot q'_c(r_-) = 4(c+1)$, so the 2-cycle is attracting for $-\frac{5}{4} < c < -\frac{3}{4}$, neutral for $c = -\frac{5}{4}$, and repelling for $c < -\frac{5}{4}$. We previously analyzed the behavior of the cycle in the neutral case; for completeness, for $g = q_c^2$ we have $g'(r_+) = -1$ and for $h = q_c^4$ we have $h''(r_+) = 0$, and $h'''(r_+) = 120(\sqrt{2} - 2) < 0$, so the neutral 2-cycle is weakly attracting.
- We can also compute the immediate basin of attraction for the fixed point p_- when $-\frac{3}{4} < c < \frac{1}{4}$ (i.e., when it is attracting).
  - There is no 2-cycle when c lies in this range, so the only possibility is for one endpoint of the basin to be p_+, since it is the only other fixed point.
  - The other solution to $q_c(x) = p_+$ is $-p_+$, so it is the other endpoint of the basin (and as expected, $-p_+$ lies to the left of p_-).
  - Thus, all points in the interval $(-p_+, p_+)$ have orbits that attract to p_-.
- In fact, even when p_- is not attracting, all of the interesting dynamics of $q_c(x)$ will happen in the interval $(-p_+, p_+)$.
  - If $x > p_+$, then $q_c(x) > x$, so the orbit of x will blow up to $+\infty$.
  - Similarly, if $x < -p_+$, then $q_c(x) > p_+$, so the orbit of $q_c(x)$, hence of x, will again blow up to $+\infty$.

2.1.2 General Properties of Bifurcations

- Among the first things we notice is that for large positive c, the quadratic family $q_c(x) = x^2 + c$ has no fixed points, and that as we decrease c, a single neutral fixed point appears at $c = \frac{1}{4}$ that splits into two points (one attracting, one repelling) for $c < \frac{1}{4}$, and these two points move apart as we continue decreasing c.
  - This is an example of a saddle-node bifurcation.
  - The word "bifurcation" means "to divide into two branches", which in this case is a reasonable description of the behavior of the fixed points as we lower the parameter c through the value $\frac{1}{4}$.
- To visualize the behavior of the fixed points as we pass through a bifurcation, we can draw a bifurcation diagram, plotting the locations of the fixed points against the parameter.
  - Here is the bifurcation diagram showing the fixed points of the quadratic family $q_c(x) = x^2 + c$:



  - It may seem peculiar to place the x-coordinate along the vertical axis, but in this case we are thinking of the parameter c as the independent variable.
- Another thing we notice about the quadratic family is that the (real-valued) 2-cycle first appears when $c < -\frac{3}{4}$, and that this is the same value of c where the fixed point p_- changes behavior from attracting to repelling. Furthermore, for $c > -\frac{3}{4}$, the fixed point p_- is attracting, but for $c < -\frac{3}{4}$ it is repelling and the 2-cycle is attracting.
  - This is an example of a period-doubling bifurcation.
- Notice that both of the bifurcations for the quadratic family $q_c(x) = x^2 + c$ occur at a neutral fixed point. In fact, this is the only time that the qualitative behavior of fixed points can change:
- Proposition (Bifurcations and Neutral Points): Suppose that $f_\lambda(x)$ is a one-parameter family and x_0, λ_0 are such that $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) \neq 1$. Then there are intervals I around x_0 and J around λ_0 and a continuously differentiable function $q : J \to I$ such that $q(\lambda_0) = x_0$ and $f_\lambda(q(\lambda)) = q(\lambda)$ and such that f_λ has no other fixed points in I. In particular, if the family f_λ has a fixed point that undergoes a bifurcation at $\lambda = \lambda_0$, then the fixed point is necessarily neutral.
  - Proof: The existence of the function q is an essentially immediate consequence of the implicit function theorem from multivariable calculus.
  - The version of the implicit function theorem we will use is as follows: if $g(y, z)$ is a function of two variables whose partial derivatives are continuous, and such that $g(y_0, z_0) = 0$ and $\frac{\partial g}{\partial z}(y_0, z_0) \neq 0$, then there exist open intervals J around y_0 and I around z_0 and a continuously differentiable function $h : J \to I$ such that $h(y_0) = z_0$ and $g(y, h(y)) = 0$ for all $y \in J$.
  - If we let $g(\lambda, x) = f_\lambda(x) - x$, then $g(\lambda_0, x_0) = f_{\lambda_0}(x_0) - x_0 = 0$ and $\frac{\partial g}{\partial x}(x_0, \lambda_0) = f'_{\lambda_0}(x_0) - 1 \neq 0$, so we can apply the theorem to g.

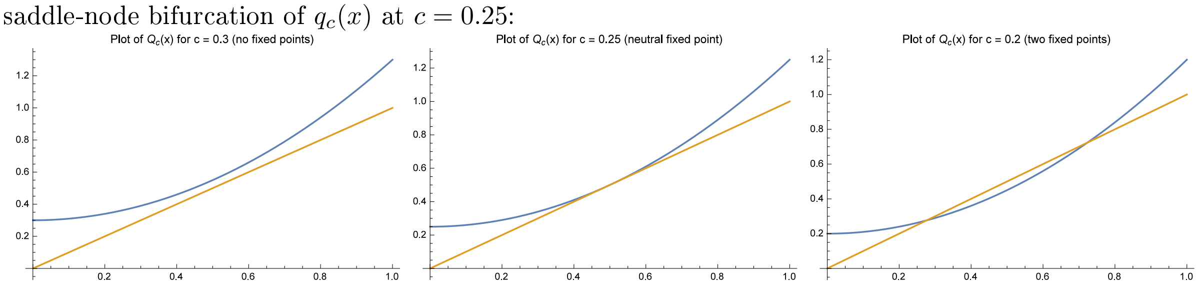
- We obtain the existence of an interval J around λ_0 and an interval I around x_0 such that there is a function $q : J \to I$ with $q(\lambda_0) = x_0$ and $g(\lambda, q(\lambda)) = 0$ for all $\lambda \in J$. The latter condition is clearly equivalent to $f_\lambda(q(\lambda)) = q(\lambda)$, so the function q has the desired properties.
- For the second statement, if x_0 lies in an attracting n-cycle for f_{λ_0}, then $\left|(f^n_{\lambda_0})'(x_0)\right| < 1$, so by the result just shown, on an open interval around λ_0, the function $f^n_{\lambda_0}$ has a unique fixed point near x_0. This means the orbit structure of the n-cycle of f containing x_0 cannot change as we move λ through λ_0. It also remains attracting near λ_0, because $|(f^n_\lambda)'(x)|$ is assumed to be continuous in x and smooth in λ, so its value will remain less than 1 in a neighborhood of (x_0, λ_0).
- Similarly, if x_0 lies in a repelling n-cycle for f_{λ_0}, the orbit structure cannot change either. Thus, if there is a change in the orbit structure of f^n_λ at $\lambda = \lambda_0$, the derivative $(f^n_{\lambda_0})'(x_0)$ must equal 1 or -1. If x_0 is a fixed point of f_{λ_0}, the chain rule gives $(f^n_{\lambda_0})'(x_0) = [f'_{\lambda_0}(x_0)]^n$, so $f'_{\lambda_0}(x_0) = \pm 1$ and thus x_0 is a neutral fixed point, as claimed.

- We will now investigate the two most common types of bifurcations that occur at a fixed point x_0: the saddle-node bifurcation (which is typically the type of bifurcation occurring when $f'_{\lambda_0}(x_0) = 1$) and the period-doubling bifurcation (which is typically the type of bifurcation occurring when $f'_{\lambda_0}(x_0) = -1$).
  - Other types of bifurcations can arise, but they are significantly less common.

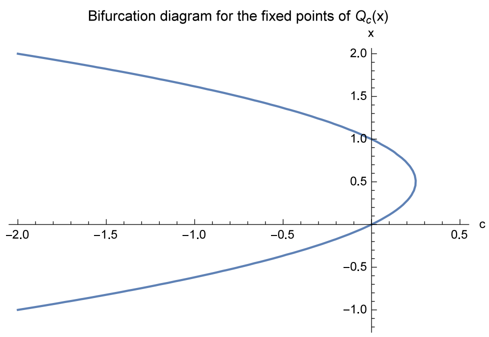
2.1.3 Saddle-Node Bifurcations

- Definition: Let $f_\lambda(x)$ be a one-parameter family of maps. We say that there is a saddle-node bifurcation (or a tangent bifurcation) at λ_0 if there is an open interval I and a positive ϵ such that
  1. For $\lambda \in (\lambda_0 - \epsilon, \lambda_0)$, the function f_λ has no fixed points in I,
  2. At $\lambda = \lambda_0$, the function f_λ has a single neutral fixed point in I, and
  3. For $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, the function f_λ has two fixed points in I: one attracting and one repelling.
  - Note: If the two intervals $(\lambda_0 - \epsilon, \lambda_0)$ and $(\lambda_0, \lambda_0 + \epsilon)$ are swapped, we also say that there is a saddle-node bifurcation at λ_0.

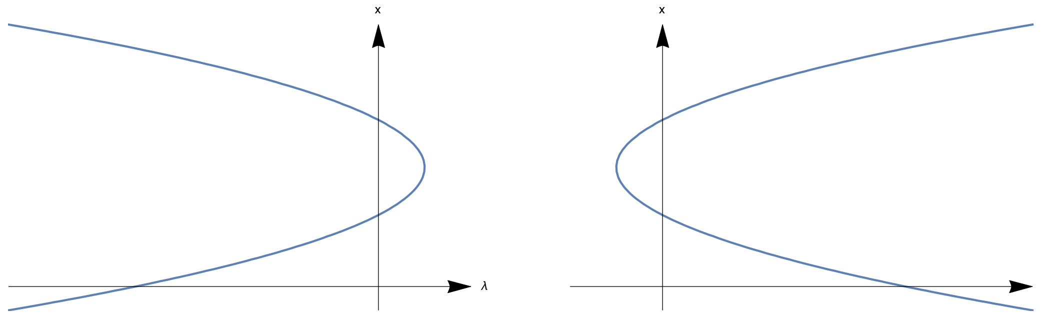
- More intuitively, a saddle-node bifurcation arises when (with λ varying) as λ passes through λ_0, a neutral fixed point for f is created that then splits into an attracting and repelling point.
  - Graphically speaking, the idea is that as λ passes through λ_0, the graph of $y = f_\lambda(x)$ moves across the line $y = x$. The value $\lambda = \lambda_0$ corresponds to the time that the graphs are tangent, and the tangency point x_0 will be the new neutral fixed point (it is necessarily neutral because the line $y = x$ is tangent to $y = f_{\lambda_0}(x)$ at $x = x_0$, meaning that $f'_{\lambda_0}(x_0) = 1$).
  - Here are the plots of $q_c(x)$ for $c = 0.3$, 0.25, and 0.2, illustrating the creation of the fixed point at the saddle-node bifurcation of $q_c(x)$ at $c = 0.25$:



  - Here (again) is the bifurcation diagram for the fixed points of the quadratic family $q_c(x) = x^2 + c$, where we can see the two branches appear and then split apart as c decreases through $\frac{1}{4}$:

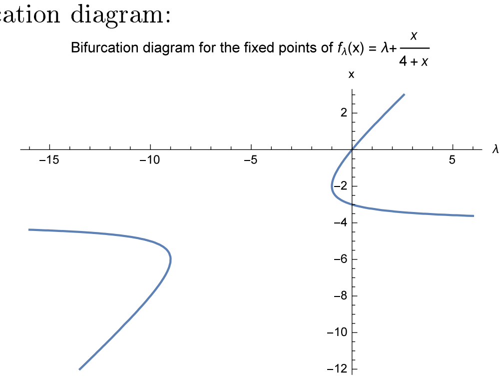


- We also remark that f^n can undergo a saddle-node bifurcation even if f does not: this corresponds to the appearance of a pair of n-cycles (one attracting, one repelling) for f as λ passes through the transitional value λ_0.
- Although it is (almost?) clear qualitatively, the definition we have given for a saddle-node bifurcation is somewhat hard to use in practice. Luckily, there is a more formal criterion:
- Theorem (Saddle-Node Bifurcation): Suppose that $f_\lambda(x)$ is a one-parameter family and x_0, λ_0 are such that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$, $f''_{\lambda_0}(x_0) \neq 0$, and $\left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$. Then there exists an open interval I around x_0 and a smooth function $p : I \to \mathbb{R}$ such that $p(x_0) = \lambda_0$, $p'(x_0) = 0$, $p''(x_0) \neq 0$, and such that $f_{p(x)}(x) = x$. In particular, f_λ has a saddle-node bifurcation at λ_0.
  - Proof: The existence of the function p is a technical calculation using the implicit function theorem.
  - Explicitly, we apply the implicit function theorem to $g(x, \lambda) = f_\lambda(x) - x$: by our hypotheses, we see that $g(x_0, \lambda_0) = 0$ and $\frac{\partial g}{\partial \lambda}(x_0, \lambda_0) \neq 0$, so the theorem applies.
  - We conclude that there exists an interval I around x_0 and a continuously differentiable function $p : I \to \mathbb{R}$ such that $p(x_0) = \lambda_0$ and $g(x, p(x)) = 0$. By the definition of g, the second condition says $f_{p(x)}(x) = x$.
  - Differentiating the relation $g(x, p(x)) = 0$ with respect to x via the multivariable chain rule yields $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial \lambda} p'(x) = 0$. Since $\frac{\partial g}{\partial x} = f'_\lambda(x) - 1$ and $\frac{\partial g}{\partial \lambda} = \frac{\partial f_\lambda}{\partial \lambda}$, setting $(x, \lambda) = (x_0, \lambda_0)$ yields $p'(x_0) = 0$, since by hypothesis $\frac{\partial g}{\partial \lambda}(x_0, \lambda_0) \neq 0$.
  - Furthermore, in general we have $p'(x) = \dfrac{f'_\lambda(x)}{\frac{\partial f_\lambda}{\partial \lambda}(x)}$, so differentiating again (and simplifying) gives $p''(x_0) = -\dfrac{f''_{\lambda_0}(x_0)}{\partial f_\lambda / \partial \lambda|_{\lambda=\lambda_0}(x_0)}$, which is nonzero since the numerator and denominator are both nonzero by hypothesis.
  - Thus, $p(x_0) = \lambda_0$, $p'(x_0) = 0$, $p''(x_0) \neq 0$, and $f_{p(x)}(x) = x$, as required.
  - Now we explain why the existence of the function p implies that f_λ has a saddle-node bifurcation: the point is that the function $\lambda = p(x)$ is implicitly expressing the location of the fixed points of $f_\lambda(x)$, since $f_{p(x)}(x) = x$.
  - The facts that $p(x_0) = \lambda_0$, $p'(x_0) = 0$, and $p''(x_0) \neq 0$ collectively say that $\lambda = p(x)$ has a local minimum or maximum value of λ_0 at x_0, meaning that the graph locally looks like a parabola with vertex at (λ_0, x_0) in one of the two possible orientations:



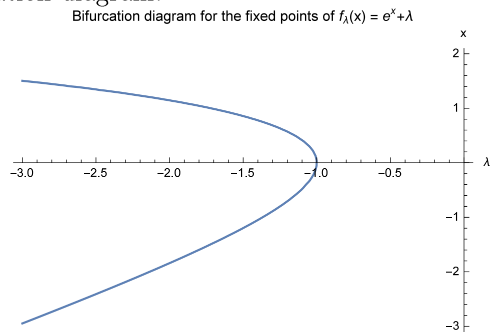
- Reading the graph horizontally rather than vertically shows that, as we vary λ, we create a fixed point that then splits into two fixed points.
- The single fixed point x_0 of $f_{\lambda_0}(x)$ is neutral since $f'_{\lambda_0}(x_0) = 1$, and when there are two fixed points, one will be attracting and the other will be repelling by the chain rule argument given above. Thus, we get a saddle-node bifurcation at $\lambda = \lambda_0$, as claimed.

- Example: Show that the one-parameter family $f_\lambda(x) = \lambda + \dfrac{x}{4+x}$ has a saddle-node bifurcation at $\lambda_0 = -1$ and at $\lambda_0 = -9$.
  - One way is simply to compute the fixed points of $f_\lambda(x)$: this requires solving $x = \lambda + \dfrac{x}{4+x}$, which is equivalent to $x^2 + (3-\lambda)x - 4\lambda = 0$, whose roots are $x = \dfrac{\lambda - 3 \pm \sqrt{9 + 10\lambda + \lambda^2}}{2} = \dfrac{\lambda - 3 \pm \sqrt{(\lambda+1)(\lambda+9)}}{2}$. Thus, if $-9 < \lambda < -1$ there are no fixed points, while $\lambda = -1, -9$ each have a single fixed point, and $\lambda > -1$ and $\lambda < -9$ have two fixed points.
  - Alternatively, we could apply the saddle-node criterion: which requires finding the value of x_0 for which $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$.
  - If $\lambda_0 = -1$ then we get $-1 + \dfrac{x_0}{4+x_0} = x_0$ and $\dfrac{4}{(x_0+4)^2} = 1$, and it is easy to see that the only solution to both equations is $x_0 = -2$. Then $f''_{\lambda_0}(x_0) = -1$ and $\dfrac{\partial f}{\partial \lambda} = 1$ (everywhere) so the criterion holds.
  - Similarly, if $\lambda_0 = -9$ then we get $-9 + \dfrac{x_0}{4+x_0} = x_0$ and $\dfrac{4}{(x_0+4)^2} = 1$, and the only solution is $x_0 = -6$. Then $f''_{\lambda_0}(x_0) = 1$ and $\dfrac{\partial f}{\partial \lambda} = 1$ (everywhere) so the criterion also holds here.
  - Here is a picture of the bifurcation diagram:



- Example: Find the value of λ_0 where there is a saddle-node bifurcation for the family $f_\lambda(x) = e^x + \lambda$.
  - For this function, it is not easy to solve for the values of the fixed points even for individual values of λ, since this requires solving the transcendental equation $x = e^x + \lambda$. (Of course, we could use Newton's method to get approximations.)
  - So we must use the saddle-node criterion, which requires finding the values of λ_0 and x_0 for which $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$.

- Since $f'_{\lambda_0}(x_0) = e^{x_0}$, the only possibility is $x_0 = 0$. Then we need $1 + \lambda_0 = f_{\lambda_0}(0) = 0$ so we must have $\lambda_0 = -1$.
- Then $f_{\lambda_0}(x_0) = 0 = x_0$, $f'_{\lambda_0}(x_0) = 1$, $f''_{\lambda_0}(x_0) = 1$, and $\dfrac{\partial f_\lambda}{\partial \lambda} = 1$ (everywhere).
- All the criteria are satisfied, so there is indeed a saddle-node bifurcation at $\lambda_0 = -1$.
- Here is a picture of the bifurcation diagram:



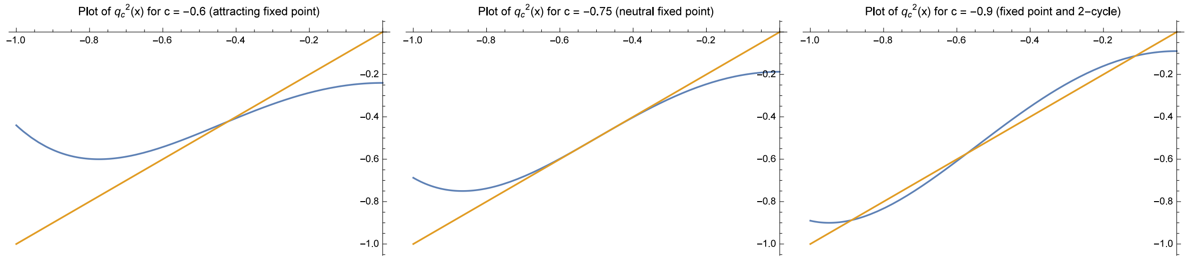
2.1.4 Period-Doubling Bifurcations

- We now analyze period-doubling bifurcations.
- Definition: Let $f_\lambda(x)$ be a one-parameter family of maps. We say that there is a period-doubling bifurcation at λ_0 if there is an open interval I and a positive ϵ such that

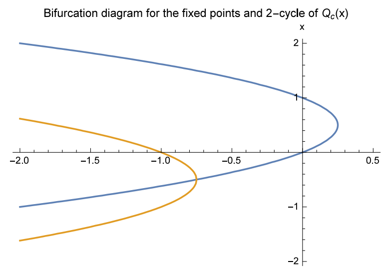
  1. For each $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ there is a unique fixed point $p_\lambda \in I$ for f_λ, whose behavior switches from attracting to repelling (or vice versa) at $\lambda = \lambda_0$,
  2. For $\lambda \in (\lambda_0 - \epsilon, \lambda_0]$, f_λ has no 2-cycles in I, and
  3. For $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$ there is a unique 2-cycle $\{q_{\lambda,1}, q_{\lambda,2}\}$ in I, which has the opposite attracting behavior to p_λ, and such that $\lim_{\lambda \to \lambda_0^+} q_{\lambda,i} = \lambda_0$ for $i = 1, 2$.

  - Note: As in the saddle-node case, if the two intervals $(\lambda_0 - \epsilon, \lambda_0)$ and $(\lambda_0, \lambda_0 + \epsilon)$ are swapped (i.e., the bifurcation occurs "backwards"), we also say that there is a saddle-node bifurcation at λ_0.

- More intuitively, a period-doubling bifurcation arises when, as λ passes through λ_0, an attracting fixed point becomes repelling and spawns an attracting 2-cycle nearby, or when a repelling fixed point becomes attracting and spawns a repelling 2-cycle nearby.
  - Note that cycles can also undergo a period-doubling bifurcation (which would correspond to a period-doubling bifurcation of a fixed point of f^n): in that case, a cycle of period n will create a new cycle of period $2n$.
  - Graphically speaking, the idea is that as λ passes through λ_0, the graph of $y = f_\lambda(x)$ is perpendicular to $y = x$ at the point of intersection. The value $\lambda = \lambda_0$ corresponds to the time that the graphs are perpendicular, and the intersection point x_0 will be the neutral fixed point in the interval (it is necessarily neutral because the line $y = -x$ is tangent to $y = f_{\lambda_0}(x)$ at $x = x_0$, meaning that $f'_{\lambda_0}(x_0) = -1$) and also the birthplace of the new 2-cycle.
  - If we plot $y = f_\lambda^2(x)$, a period-doubling bifurcation will occur when the graph "twists through" the line $y = x$, transitioning from having a single intersection point (corresponding to the fixed point) to having three intersection points (the two points on the 2-cycle along with the fixed point).
  - Here are the plots of $q_c^2(x)$ for $c = -0.6, -0.75$, and -0.9, illustrating the creation of the 2-cycle at the period-doubling bifurcation of $q_c(x)$ at $c = -0.75$:



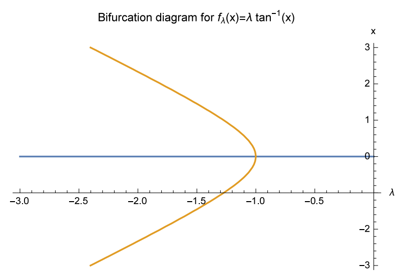
- Like with saddle-node bifurcations, we can visualize period-doubling bifurcations with a bifurcation diagram, but we need to draw the fixed points for f^2 rather than f.
  - Here is the bifurcation diagram for the quadratic family, with the two curves representing fixed points and 2-cycles in different colors:



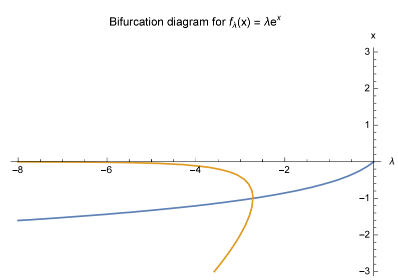
  - We can see quite plainly how the curve representing the 2-cycle "sprouts" from the fixed-point curve at $c = -0.75$.
- Like with the saddle-node bifurcation, there is an analytic criterion for the presence of a period-doubling bifurcation.
- <u>Theorem</u> (Period-Doubling Bifurcation): Suppose that $f_\lambda(x)$ is a one-parameter family and x_0, λ_0 are such that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = -1$, and $\left.\dfrac{\partial (f_\lambda^2)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$. Then there exists an open interval I around x_0 and a smooth function $p : I \to \mathbb{R}$ such that $f_{p(x)}(x) \neq x$ but $f^2_{p(x)}(x) = x$. In particular, f_λ has a period-doubling bifurcation at λ_0.
  - <u>Proof</u>: Like with the saddle-node theorem, the proof is essentially an application of the implicit function theorem.
  - Earlier, we showed that if $f_\lambda(x)$ is a one-parameter family and x_0, λ_0 are such that $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) \neq 1$, then there are intervals I around x_0 and J around λ_0 and a continuously differentiable function $q : J \to I$ such that $q(\lambda_0) = x_0$ and $f_\lambda(q(\lambda)) = q(\lambda)$ and such that f_λ has no other fixed points in I.
  - The hypotheses obviously apply to f in our case, so now let $g_\lambda(x) = f_\lambda(x + q(\lambda)) - q(\lambda)$. It is easy to see that $f_\lambda(x) = g_\lambda(x - q(\lambda)) + q(\lambda)$, so for any fixed λ, the orbit behavior of f will be the same as the orbit behavior of g.
  - It is then a fairly straightforward verification that the given hypotheses on f are equivalent to the conditions $g_{\lambda_0}(0) = 0$, $g'_{\lambda_0}(0) = -1$, and $\left.\dfrac{\partial (g_\lambda^2)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(0) \neq 0$ on g.
  - Moreover, the condition $f_\lambda(q(\lambda)) = q(\lambda)$ becomes $g_\lambda(0) = 0$ for all λ in an interval around λ_0, and we are also given that there are no other fixed points for g_λ near 0.
  - Thus, we have arranged matters so that the fixed point of g_λ remains stationary at 0 as we vary λ. Our goal now is to show that a nonzero 2-cycle will arise for g_λ.
  - We would like to apply the implicit function theorem to $G(x, \lambda) = g_\lambda^2(x) - x$ to try to show that a 2-cycle exists. However, this will not work because the derivative $\dfrac{dG}{dx}(0, \lambda_0) = g'_\lambda(0)^2 - 1 = 0$ fails the hypothesis of the test.

- We instead apply the theorem to the function $h(x, \lambda) = \dfrac{G(x, \lambda)}{x}$, where we define $h(0, \lambda) = \dfrac{\partial G}{\partial x}(0, \lambda)$ so as to make it continuous at $x = 0$.
- Then $h(0, \lambda_0) = 0$ from above, and also $\dfrac{\partial h}{\partial \lambda}(0, \lambda) = \lim_{x \to 0} \dfrac{\partial}{\partial \lambda}\left[\dfrac{g_\lambda^2(x)}{x}\right] = \dfrac{\partial (g_\lambda^2)'}{\partial \lambda}\Big|_{\lambda=\lambda_0}(0)$, which is nonzero by hypothesis.
- So we can apply the implicit function theorem to obtain an open interval I around x_0 and a smooth function $p : I \to \mathbb{R}$ such that $p(0) = \lambda_0$ and $h(x, p(x)) = 0$. Thus, we see that $g_{p(x)}^2(x) = x$, and also by our construction x is not a fixed point of $g_{p(x)}$ since the only fixed point near 0 is 0: thus, x has period exactly 2 for $g_{p(x)}$.
- Finally, it is a straightforward (though messy) computation with the chain rule to compute the attracting/repelling behavior of the 2-cycle and verify that it has the opposite behavior to the fixed point at 0 for g_λ. (We omit the details.)

- Example: Show that there is a period-doubling bifurcation for $f_\lambda(x) = \lambda \tan^{-1}(x)$ at $\lambda_0 = -1$.
  - To apply the criterion, we want to find x_0 such that $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) = -1$, and then check that $\dfrac{\partial (f_\lambda^2)'}{\partial \lambda}\Big|_{\lambda=\lambda_0}(x_0) \neq 0$.
  - We have $f_{\lambda_0}(x) = -\tan^{-1}(x)$ and $f'_{\lambda_0}(x) = \dfrac{-1}{1+x^2}$ so the only possibility is to have $x_0 = 0$, which does satisfy both equations.
  - Also, $f_\lambda^2(x) = \lambda \tan^{-1}(\lambda \tan^{-1}(x))$, so $(f_\lambda^2)'(0) = \lambda^2$ and thus the desired derivative is $2\lambda \neq 0$.
  - Thus, there is a period-doubling bifurcation at $\lambda_0 = -1$.
  - Here is a plot of the bifurcation diagram, demonstrating the creation of the 2-cycle at $\lambda_0 = -1$ from the fixed point $x_0 = 0$:



- Example: Find the value of λ_0 where there is a period-doubling bifurcation for $f_\lambda(x) = \lambda e^x$.
  - To apply the criterion, we want to find x_0 and λ_0 such that $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) = -1$, and then check that $\dfrac{\partial (f_\lambda^2)'}{\partial \lambda}\Big|_{\lambda=\lambda_0}(x_0) \neq 0$.
  - We have $x_0 = \lambda_0 e^{x_0}$ and $-1 = \lambda_0 e^{x_0}$ so the only possibility is to have $x_0 = -1$, and then $\lambda_0 = -e$.
  - Also, $f_\lambda^2(x) = \lambda e^{\lambda e^x}$, so $(f_\lambda^2)'(0) = \lambda^2 e^\lambda$ and thus the desired derivative is $\lambda^2 e^\lambda + 2\lambda e^\lambda$ which is nonzero at $\lambda = -e$.
  - Thus, there is a period-doubling bifurcation at $\lambda_0 = -1$.
  - Here is a plot of the bifurcation diagram, demonstrating the creation of the 2-cycle at $\lambda_0 = -1$ from the fixed point $x_0 = 0$:



2.2 Critical Orbits and Attracting Cycles

- In this section we will develop some computational tools to study attracting cycles. We begin by defining the Schwarzian derivative, which is a quite opaque and mysterious construction. Our main goal is to show that a function with negative Schwarzian derivative (a statement that applies to many polynomials) has the property that most attracting cycles will attract a critical point.
- We will then use these results to study attracting cycles of one-parameter families.

2.2.1 The Schwarzian Derivative

- The Schwarzian derivative is a rather strange tool, and rather than trying to motivate its origins further, we will simply define it and then discuss some of its uses.
- Definition: If f''' exists, the Schwarzian derivative $Sf(x)$ is defined to be $Sf(x) = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$.
  - In the examples below we will suppress the arguments of the functions for conciseness.
  - Example: If $f = x^2 + x$, then $f' = 2x + 1$, $f'' = 2$, and $f''' = 0$, so $Sf = \boxed{-\dfrac{3}{2(2x+1)^2}}$.
  - Example: If $f = x^3 - x$, then $f' = 3x^2 - 1$, $f'' = 6x$, and $f''' = 6$, so $Sf = \boxed{-\dfrac{6+36x^2}{(3x^2-1)^2}}$.
  - Example: If $f = x^4 + 1$, then $f' = 4x^3$, $f'' = 12x^2$, and $f''' = 24x$, so $Sf = \boxed{-\dfrac{15}{2x^2}}$.
  - Example: If $f = e^{2x}$, then $f' = 2e^{2x}$, $f'' = 4e^{2x}$, and $f''' = 8e^{2x}$, so $Sf = \boxed{-2}$ (identically).
  - Example: If $f = x^3 + x$, then $f' = 3x^2 + 1$, $f'' = 6x$, and $f''' = 6$, so $Sf = \boxed{\dfrac{6-36x^2}{(3x^2+1)^2}}$.
- Notice that for the first three polynomials we gave, the Schwarzian derivative was always negative (where we adopt the natural convention of declaring that $f(c) = -\infty$ if $\lim_{x\to c} f(x) = -\infty$, and considering $-\infty$ to be a negative number). This is a common feature of polynomials:
- Proposition (Schwarzian Derivatives of Polynomials): If p is a polynomial of degree ≥ 2 such that all roots of p' are real, then $Sp < 0$. In particular, if all roots of p are real, then $Sp < 0$.
  - Proof: Suppose first that all roots of p' are real and write $p' = c(x - r_1)\cdots(x - r_n)$ for real numbers r_i, where by the hypothesis on the degree, $n \geq 1$.
  - Then $\ln(p'/c) = \sum_{i=1}^{n} \ln(x - r_i)$, so differentiating both sides gives $\dfrac{p''}{p'} = \sum_{i=1}^{n} \dfrac{1}{x - r_i}$.
  - Differentiating again produces $\dfrac{p'''}{p'} - \left(\dfrac{p''}{p'}\right)^2 = \dfrac{p'''p' - (p'')^2}{(p')^2} = -\sum_{i=1}^{n} \dfrac{1}{(x-r_i)^2}$.
  - Applying the mean value theorem to p on (s_i, s_{i+1}) shows that p' has a root in this interval for each $1 \leq i \leq k-1$. Also, p' is clearly divisible by $(x - s_i)^{a_i - 1}$ for each i. Then p' has at least $(k-1) + \sum_{i=1}^{k}(a_i - 1) = d - 1$ real roots (counted with multiplicity), but since p' has degree $d - 1$, all of its roots must be real.
  - Thus, $Sp = \dfrac{p'''}{p'} - \left(\dfrac{p''}{p'}\right)^2 - \dfrac{1}{2}\left(\dfrac{p''}{p'}\right)^2 = -\sum_{i=1}^{n} \dfrac{1}{(x-r_i)^2} - \dfrac{1}{2}\left(\sum_{i=1}^{n} \dfrac{1}{x - r_i}\right)^2 < 0$, since the second term is nonnegative while the first term is always negative.

- For the second statement, we will show that if all roots of p are real, then all roots of p' are real. Suppose p has degree d and that the distinct roots of p are $s_1 < s_2 < \cdots < s_k$ where s_i is a root having multiplicity a_i for $1 \le i \le k$: thus, $\sum_{i=1}^{k} a_i = d$.
- Applying the mean value theorem to p on (s_i, s_{i+1}) shows that p' has a root in this interval for each $1 \le i \le k-1$. Also, p' is clearly divisible by $(x-s_i)^{a_i-1}$ for each i. Then p' has at least $(k-1)+\sum_{i=1}^{k}(a_i-1) = d-1$ real roots (counted with multiplicity), but since p' has degree $d-1$, all of its roots must be real.

- Although the Schwarzian derivative does not interact particularly well with most operations on functions (e.g., addition, multiplication, division) it does behave somewhat reasonably with respect to composition:
- Proposition (Schwarzian Chain Rule): For any f and g, if $h = f \circ g$, then $Sh(x) = Sf(g(x)) \cdot g'(x)^2 + Sg(x)$. In particular, If Sf and Sg are both always negative, then Sh is also always negative.
  - Proof: For the first statement, the usual chain rule gives

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ h''(x) &= f''(g(x)) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x) \\ h'''(x) &= f'''(g(x)) \cdot g'(x)^3 + 3f''(g(x)) \cdot g''(x) \cdot g'(x) + f'(g(x)) \cdot g'''(x) \end{aligned}$$

from which we obtain

$$\begin{aligned} Sh(x) &= \frac{f'''(g(x)) \cdot g'(x)^3 + 3f''(g(x)) \cdot g''(x) \cdot g'(x) + f'(g(x)) \cdot g'''(x)}{f'(g(x)) \cdot g'(x)} - \frac{3}{2}\left(\frac{f''(g(x)) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x)}{f'(g(x)) \cdot g'(x)}\right)^2 \\ &= \frac{f'''(g(x))}{f'(g(x))} g'(x)^2 - \frac{3}{2}\left(\frac{f''(g(x))}{f'(g(x))}\right)^2 g'(x)^2 + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 \\ &= Sf(g(x)) \cdot g'(x)^2 + Sg(x) \end{aligned}$$

where the middle term of the two fractions in the first line cancel.
  - The second statement follows immediately from the first, since by hypothesis $Sf(g(x)) \cdot g'(x)^2 \le 0$ and $Sg(x) < 0$.
  - Remark: We have included this routine computation because it sheds a little light on why the coefficient $-3/2$ shows up in the definition of the Schwarzian derivative: it is the only constant that would make the cancellation work.

- A geometric interpretation of the Schwarzian derivative is hard to come by. However, if the Schwarzian derivative of a function is always negative, then we can at least say something about the behavior of its derivative:
- Proposition (Schwarzian Min-Max Principle): Suppose that Sf is always negative. Then f' cannot have a positive local minimum or a negative local maximum.
  - Proof: Suppose that x_0 is a local extremum of f' (i.e., a local minimum or maximum). Then by standard calculus we know that $f''(x_0) = 0$, so $Sf(x_0) = \dfrac{f'''(x_0)}{f'(x_0)}$ is negative.
  - If f' has a positive local minimum at x_0, by definition we get $f'(x_0) > 0$, so it must be the case that $f'''(x_0) < 0$. But this means f'' changes sign from positive to negative at x_0, which would imply that x_0 is a maximum for f' rather than a minimum (contradiction).
  - Similarly, if f' has a negative local maximum at x_0, we get instead $f'(x_0) < 0$ so that $f'''(x_0) > 0$. But this means f'' changes sign from negative to positive at x_0, meaning that x_0 is a minimum rather than a maximum.

- One of the key properties of a function with a negative Schwarzian derivative is that most attracting cycles will attract a critical point.

- Recall that if f is differentiable, then x_0 is a critical point of f if $f'(x_0) = 0$.

- Theorem (Critical Orbits): Suppose that Sf is always negative. If $\{x_1, \cdots, x_n\}$ is a (weakly) attracting periodic cycle of f, then either the immediate attracting basin of some x_i is either infinite or contains a critical point of f. In particular, if f has k critical points, then it has at most $k+2$ attracting cycles.

  - Proof: We will first show that if x_0 is a (weakly) attracting fixed point with finite immediate attracting basin, then the basin contains a critical point.
  - By our earlier analysis, we know that if I is finite then $I = (a, b)$ where each of $f(a)$ and $f(b)$ are either a or b.
  - If $f(a) = f(b)$, then by the mean value theorem, f' has a zero in (a, b) and thus (a, b) contains a critical point.
  - Next, suppose $f(a) = a$ and $f(b) = b$. Consider $f(x) - x$ on the interval (a, x_0): it can never equal zero since that would give a fixed point of f (which cannot lie in the attracting basin of x_0). Since x_0 is (weakly) attracting, we must therefore have $f(x) > x$ on (a, x_0), as otherwise nearby orbits would move away from x_0. Similarly, we must have $f(x) < x$ on (x_0, b).
  - Now, the mean value theorem implies that there exists a $c \in (a, x_0)$ such that $f'(c) = \dfrac{f(a) - f(x_0)}{a - x_0} = 1$, and similarly there exists a $d \in (x_0, b)$ such that $f'(d) = 1$. Therefore, on (c, d), we have $f'(c) = 1$, $f'(x_0) < 1$, and $f'(d) = 1$, so f' has a local minimum somewhere in (c, d). Now by the Schwarzian min-max principle, f' cannot have a positive local minimum in (c, d) so it must attain a negative value. By the intermediate value theorem, it must take the value 0, so f has a critical point in (a, b).
  - Finally, suppose $f(a) = b$ and $f(b) = a$. If we let $g = f^2$, then x_0 is (weakly) attracting with immediate basin I for g, $g(a) = a$, $g(b) = b$, and by the chain rule for Schwarzian derivatives, $Sg < 0$. Thus, by the previous case, g has a critical point y in (a, b). But since $g'(y) = f'(f(y)) \cdot f'(y)$, it follows that one of y and $f(y)$ is a critical point of f. But they both lie in (a, b) by the definition of the immediate basin, so either way f has a critical point in (a, b).
  - Now we extend the previous result to n-cycles.
  - Suppose that $\{x_1, \cdots, x_n\}$ is an attracting n-cycle for f. If the immediate basin for any of the points as attracting fixed points for f^n is infinite, we are done.
  - Otherwise, if they are all finite, then by the above argument, the immediate basin for x_1 contains a critical point y of f^n.
  - But by the chain rule, $(f^n)'(y) = f'(f^{n-1}(y)) \cdot f'(f^{n-2}(y)) \cdots f'(y)$, so $f^i(y)$ must be a critical point of f for some $0 \le i \le n-1$.
  - But $f^i(y)$ lies in the attracting basin of x_{i+1}: by assumption, since $f^{nk}(y) \to x_1$, applying f^i and the continuity of f yields $f^i(f^{nk}(y)) = f^{nk}(f^i(y)) \to f^i(x_1) = x_{i+1}$. Thus, the n-cycle $\{x_1, \cdots, x_n\}$ attracts a critical point of f.
  - For the last statement, at most 2 attracting orbits of f can have an immediate basin that extends to ∞. Each finite attracting orbit must attract at least one critical point of f, so if f has k critical points, it can have at most $k+2$ attracting cycles.

2.2.2 Numerical Computation of Attracting Cycles

- Our theorem showing that the attracting cycles of a function with negative Schwarzian derivative must attract a critical point (or extend to ∞) is quite computationally useful in studying the orbit behavior of polynomial maps. More explicitly:

- Proposition: If $p(x)$ is a polynomial of degree $d \ge 2$ whose Schwarzian derivative is always negative (in particular, if the roots of p or p' are all real), then every attracting cycle of p attracts a critical point of p. In particular, p has at most $d - 1$ attracting cycles.

  - Proof: We showed earlier that a polynomial with real roots and degree at least 2 has negative Schwarzian derivative. If $Sp < 0$, then by the theorem on critical orbits, every attracting cycle of p either has an infinite immediate basin or attracts a critical point of p. We just need to eliminate the first possibility.

- We start by showing that for any polynomial $q(x)$ of degree at least 2, all sufficiently large initial points have an orbit that blows up to $\pm\infty$.
- So let $q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 2$ and $a_n \neq 0$, and let $R = \sum_{i=0}^{n-1} |a_i|$.
- We claim that if $|x| \geq \max(1,\, 2R\,|a_n|\,,\, (4/\,|a_n|)^{1/(n-1)})$, then $|q(x)| \geq 2\,|x|$. By iterating this result we see immediately that sufficiently large orbits will tend to ∞.
- So suppose $|x| \geq \max(1,\, 2R\,|a_n|\,,\, (4/\,|a_n|)^{1/(n-1)})$. Then we get

$$\begin{aligned} |q(x)| &\geq |a_n x^n| - \left[\left|a_{n-1}x^{n-1}\right| + \cdots + |a_1 x| + |a_0|\right] \\ &\geq |a_n x^n| - [|a_{n-1}| + \cdots + |a_1| + |a_0|]\left|x^{n-1}\right| \\ &= |x^n| \cdot [|a_n| - |R|\,/\,|x|] \\ &\geq |x| \cdot \left|x^{n-1}\right| \cdot \frac{1}{2}\,|a_n| \\ &\geq |x| \cdot 2 \end{aligned}$$

  where the four inequality steps respectively follow from $n-1$ applications of the triangle inequality, the assumption that $|x| \geq 1$, the assumption that $|x| \geq 2R\,|a_n|$, and the assumption that $\left|x^{n-1}\right| > 4/\,|a_n|$.
- Now observe that p^n is a polynomial of degree ≥ 2 if p is, for any n. The previous observation applied to $q = p^n$ shows that sufficiently large orbits of p^n will always tend to $\pm\infty$, so p^n cannot have an attracting fixed point whose immediate basin extends to ∞ or $-\infty$. Thus, p cannot have an attracting n-cycle whose immediate basin extends to ∞, so every attracting cycle attracts a critical point.
- If p has degree d, then since there are (at most) $d-1$ critical points of p, there are at most $d-1$ attracting cycles.

- The proposition suggests that if we want to find attracting cycles of a polynomial, all we need to do is numerically compute the orbits of the critical points, then look to see whether they seem to be attracted to a cycle.
  - Computing critical orbits is not a "magic bullet", however: if the attracting cycle has a large number of points, or the polynomial has complicated orbit behavior, it may be not be possible to compute with enough accuracy to find the attracting cycle.
  - Of course, a function need not have any attracting cycles at all, in which case the orbit of any critical point will never settle down towards a cycle.
  - Even if there is an attracting cycle, if it is sufficiently complicated then we may be unable to tell the difference between "extremely lengthy attracting cycle" and "no attracting cycle".
- If we can find what appears to be an attracting cycle, then it is usually straightforward to prove its existence:
- Proposition (Computing Cycles): Suppose f is continuously differentiable and there exist disjoint open intervals $I_1, I_2, \ldots, I_n$ such that no interval I_i contains a critical point of f, that $f(I_i) \subseteq I_{i+1}$ for $1 \leq i \leq n-1$, and that $f(I_n)$ is strictly contained in I_1. Then I_1 contains a periodic point of exact period n for f. Furthermore, the periodic point is attracting provided that none of the I_i contain a critical point of f', and that the product $\prod_{i=1}^{n} \max(|f'(a_i)|\,, |f'(b_i)|)$ is less than 1, where $I_i = (a_i, b_i)$.
  - Remark: There is an analogous way to show the existence of a repelling cycle: simply change the interval statements to $f(I_i) \supseteq I_{i+1}$ and the derivative statement to $\prod_{i=1}^{n} \min(|f'(a_i)|\,, |f'(b_i)|) > 1$. Then essentially the same proof as below will show the existence of a repelling n-cycle for f.
  - Proof: Suppose we have intervals I_i satisfying the requirements.
  - We first observe that if $J = (c, d)$ is any interval that does not contain a critical point of f, then f' is continuous and never zero on J (else J would contain a critical point of f), so f is either monotone increasing or monotone decreasing there. In particular, $f(J)$ is the open interval with endpoints $f(c)$, $f(d)$.

- Now let $I_1 = (a, b)$. By the given assumptions, $f^n(I_1) \subseteq f^{n-1}(I_2) \subseteq \cdots \subseteq f(I_n) \subset I$ so $f^n(I_1)$ is strictly contained in I_1. Then by the Brouwer fixed point theorem, I_1 contains a fixed point x_0 of f^n. Furthermore, because none of the other I_i intersect I_1, this fixed point cannot have period less than n for f, so it has exact period n for f.
- For the last part, if an interval J does not contain a critical point of f' then f' is monotone on J by the argument given above: so if no I_i contains a critical point of f', then f' is monotone on each I_i.
- Therefore, since $f^i(x_0)$ is contained in $I_i = (a_i, b_i)$, by the chain rule and the monotonicity of f' on each interval, we have

$$|(f^n)'(x_0)| = \prod_{i=1}^{n} |f'(f^i(x_0))| \le \prod_{i=1}^{n} \max_{x \in I_i} |f'(x)| = \prod_{i=1}^{n} \max(|f'(a_i)|, |f'(b_i)|) < 1$$

and thus x_0 is attracting as claimed.

- The way we apply the proposition to show the existence of an attracting n-cycle is by taking I_1 to be a very small interval around one point on the n-cycle and I_k to be the kth iterate of I_1 (rounded outwards).
  - If I_1 is small enough, then its iterates should avoid critical points of f and f', so all of the quantities are easy to compute.
  - We phrased the proposition using intervals in order to build in a way to account for roundoff errors during the computations while minimizing the number of quantities that need to be computed. The only quantities that need to be evaluated are the endpoints of all the intervals, the critical points of f and f' (easily found using Newton's method), and the value of f' at the endpoint of each interval.
  - It is easy to program a computer to evaluate everything to the necessary accuracy, but we will give an example of how to do it "by hand".

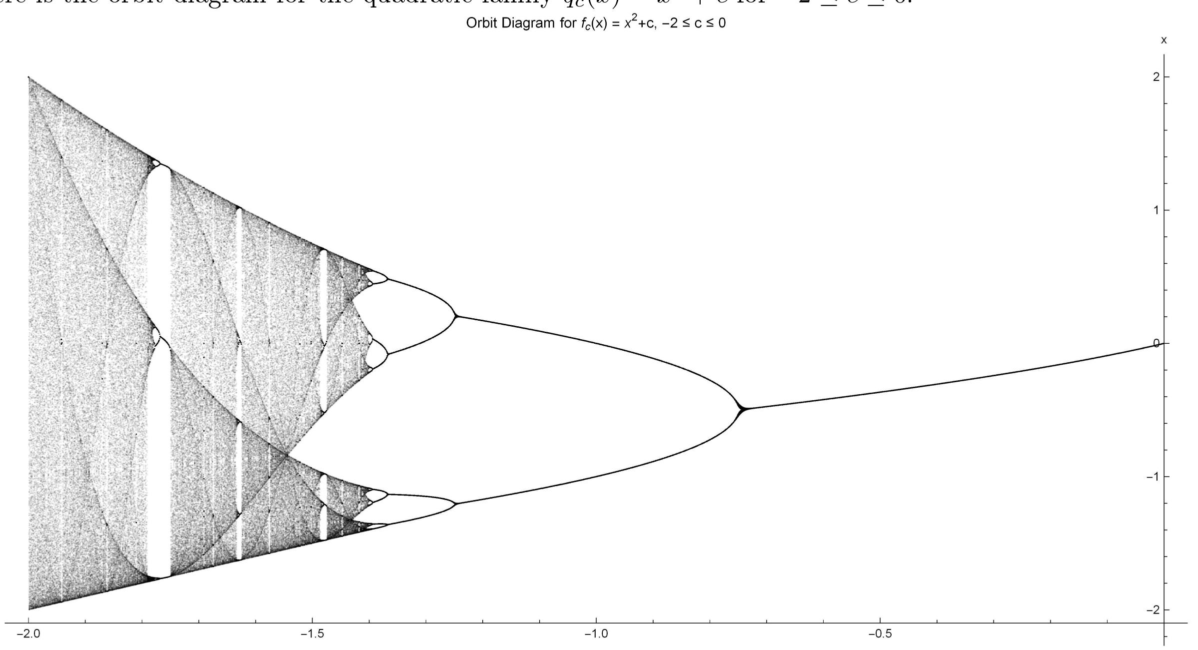
- Example: Show that the logistic map $p_{3.74}(x) = 3.74x(1-x)$ has an attracting 5-cycle and compute the values on it to three decimal places.
  - The critical point of $p_{3.74}$ is $x = \frac{1}{2}$. To 6 decimal places, the 100th through 110th elements in the orbit of the critical point are 0.496176, 0.934945, 0.227476, 0.657234, 0.842538, 0.496176, 0.934945, 0.227476, 0.657234, 0.842538. These seem fairly stable.
  - Since the critical point is $x = \frac{1}{2}$ and there are no critical points of f', f will be monotone on the intervals we choose.
  - We will try taking the interval $I_1 = (0.2274, 0.2276)$.
  - We compute $f(I_1) = (0.6570777,\ 0.6574855)$ to seven decimal places, so rounding outward allows us to take $I_2 = (0.657077,\ 0.657486)$.
  - Then $f(I_2) = (0.8422411,\ 0.8427223)$ so we take $I_3 = (0.842241,\ 0.842723)$.
  - Then $f(I_3) = (0.4957031,\ 0.4969380)$ so we take $I_4 = (0.495703,\ 0.496938)$.
  - Then $f(I_4) = (0.9349309,\ 0.9349650)$ so we take $I_5 = (0.934930,\ 0.934965)$.
  - Finally, $f(I_5) = (0.2274124,\ 0.2275262)$ which does indeed lie inside I_1.
  - Thus, there is a 5-cycle for $p_{3.74}$ whose elements lie in the five intervals I_1 through I_5.
  - So, to 3 decimal places, we conclude that the values are $\boxed{\{0.227,\ 0.657,\ 0.842,\ 0.496,\ 0.935\}}$.
  - We can also compute the larger magnitude of $p'_{3.74}(x)$ at each endpoint of the five intervals.
  - The bigger value in each case is 2.03905, −1.17800, −2.56357, 0.03215, and −3.25354. Since the product of these values−0.64410 has absolute value less than 1, the cycle is $\boxed{\text{attracting}}$, as claimed.

2.3 Orbit Diagrams

- We now turn to the problem of studying attracting orbits in one-parameter families.
  - Most everything in this section will be computational and qualitative: we will make a number of observations based on experimental evidence and our earlier results. (The few things we do prove will be rather numerical in nature, and based off of our computations.)
  - Our aim, ultimately, is to provide motivation for the definition of chaotic dynamical systems.
- If the one-parameter family f_λ is such that $Sf_\lambda < 0$ for every value of λ, then our results indicate that we can try numerically computing the asymptotic orbit (i.e., the long-term orbit behavior) of each critical point of f_λ for each value of λ.
  - To do this, we can compute the first 500 or so iterates of the critical point and then throw away the first few iterates (certainly 200 or so is sufficient, to ensure that we will presumably be seeing long-term behavior).
  - The exact numbers will, of course, change the results slightly, but not substantially.
- We can package all of this information in an orbit diagram: we plot a large number of points in the asymptotic orbit of each critical point of the function f_λ against the parameter λ, for a large number of λ.
  - The orbit diagram has the same basic setup as a bifurcation diagram: the parameter λ goes on the horizontal axis and the values of x representing points in a critical orbit go on the vertical axis.

2.3.1 The Quadratic Family $q_c(x) = x^2 + c$

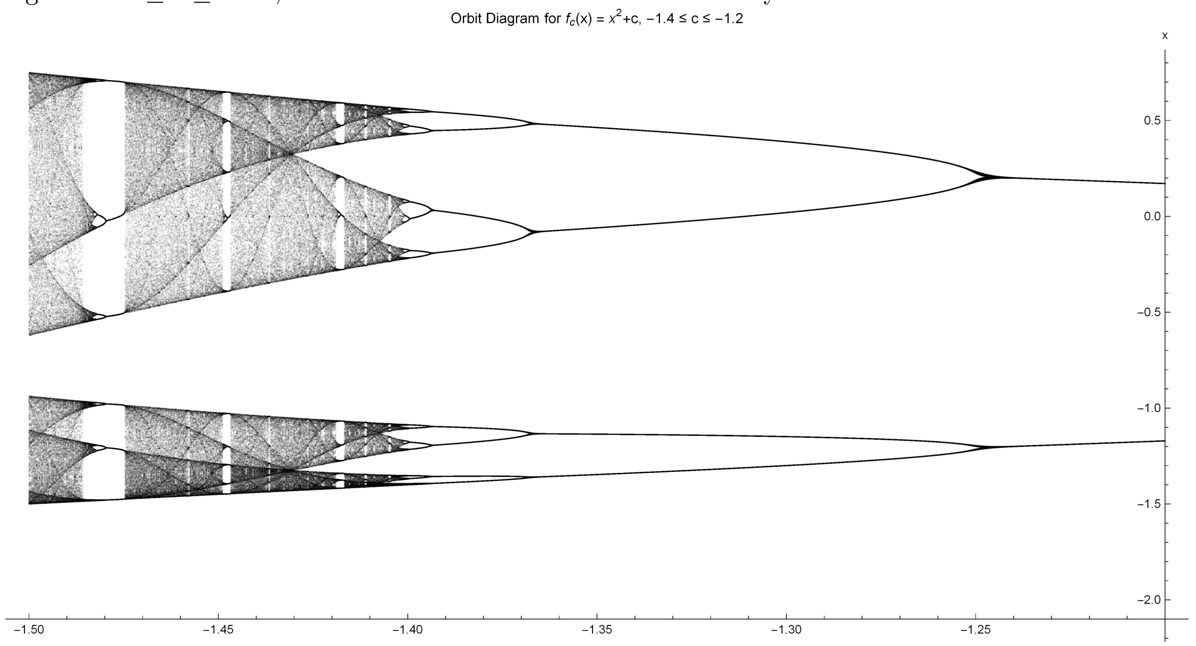
- Here is the orbit diagram for the quadratic family $q_c(x) = x^2 + c$ for $-2 \le c \le 0$:



  - The picture above was created by evaluating the 200th through 500th elements of the orbit of 0 under $f_c(x)$ for 2000 equally-spaced values of c in $[-2, 0)$.
  - Increasing the number of orbit elements computed, the numerical precision of those computations, or the number of values of c will alter the picture, but not substantially. (The reader is invited to experiment with the parameters.)
- Observation 0: For each value of c, there is at most one attracting cycle for $f_c(x)$. Furthermore, if $c < -2$, then the critical orbit diverges to $+\infty$.

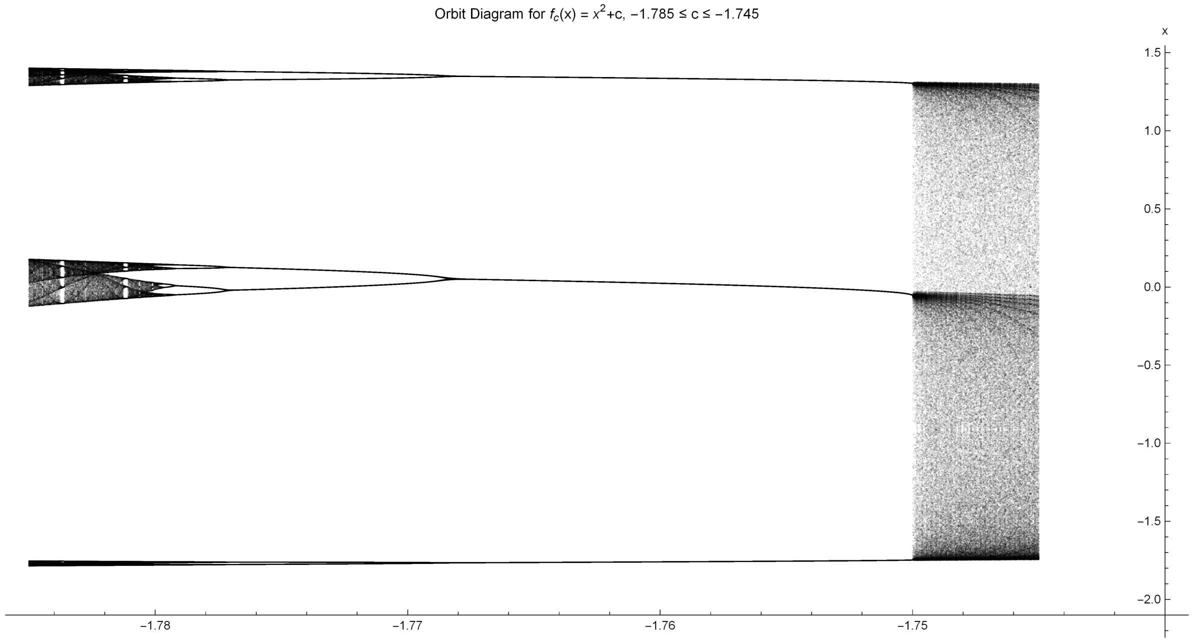
- ◦ In fact, we have already shown the first statement: since $f_c'(x) = 2x$ has all its roots real and f_c is a polynomial of degree 2, it can have at most 1 attracting cycle.
- ◦ For the second statement, observe that $f_c(0) = c$, $f_c^2(0) = c^2+c > -c$, and if $x > -c > 2$, then $f_c(x) > x$: thus, the critical orbit will go to $+\infty$. (This is why we stop plotting the orbit diagram at $c = -2$.)

- <u>Observation 1</u>: As c decreases through roughly $c = 1.4$, there appear to be a series of period-doubling bifurcations, yielding attracting cycles of periods 1, 2, 4, 8, 16, ...
  - ◦ These period-doubling bifurcations occur when the graph "forks". If we narrow our attention to the region $-1.5 \le c \le -1.2$, we can see these bifurcations more clearly:

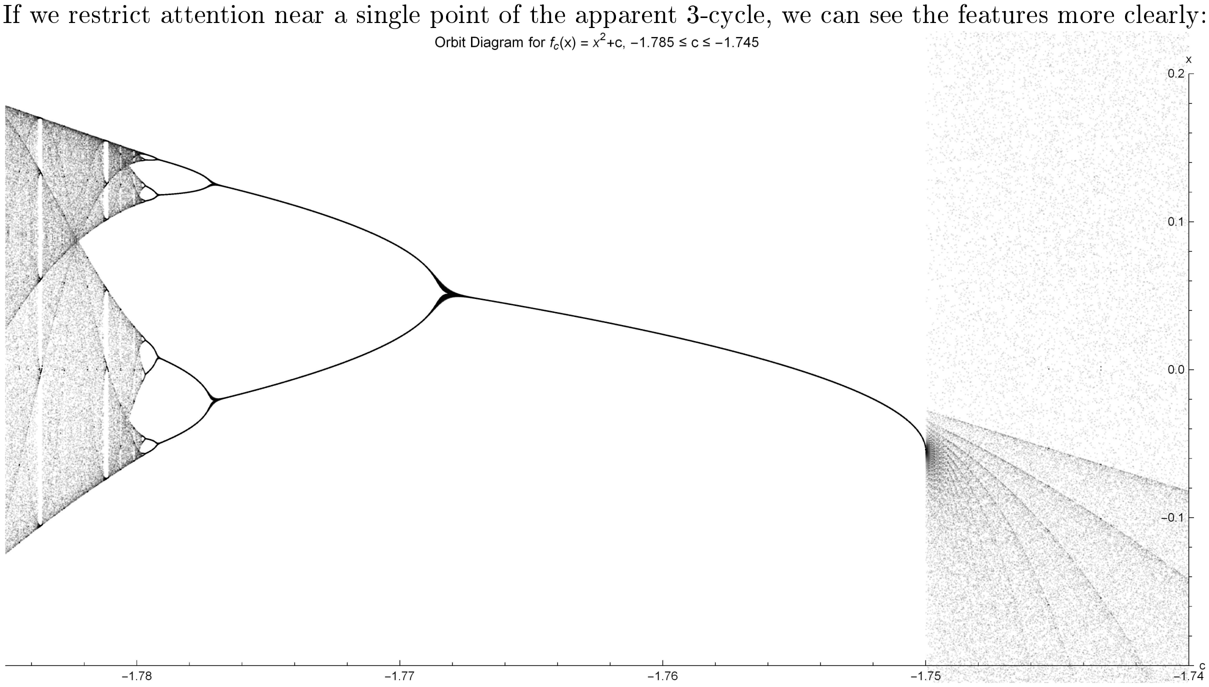


- <u>Observation 2</u>: There appear to be many values of the parameter c where there is no attracting cycle, and for which the critical orbit seems "chaotic".
  - ◦ For c near the value -1.55 (or so) the orbit diagram is essentially a solid black vertical strip, suggesting that the critical orbit essentially fills an interval, in the sense that there are no obvious "gaps".

- <u>Observation 3</u>: There appear to be "windows of stability" where f_c is no longer chaotic, and an attracting cycle reappears. In each window, as c decreases, the attracting cycle seems to undergo a series of period-doubling bifurcations until the map once again becomes "chaotic".
  - ◦ For example, there is a comparatively large window for c approximately equal to -1.75, where an attracting 3-cycle seems to arise:



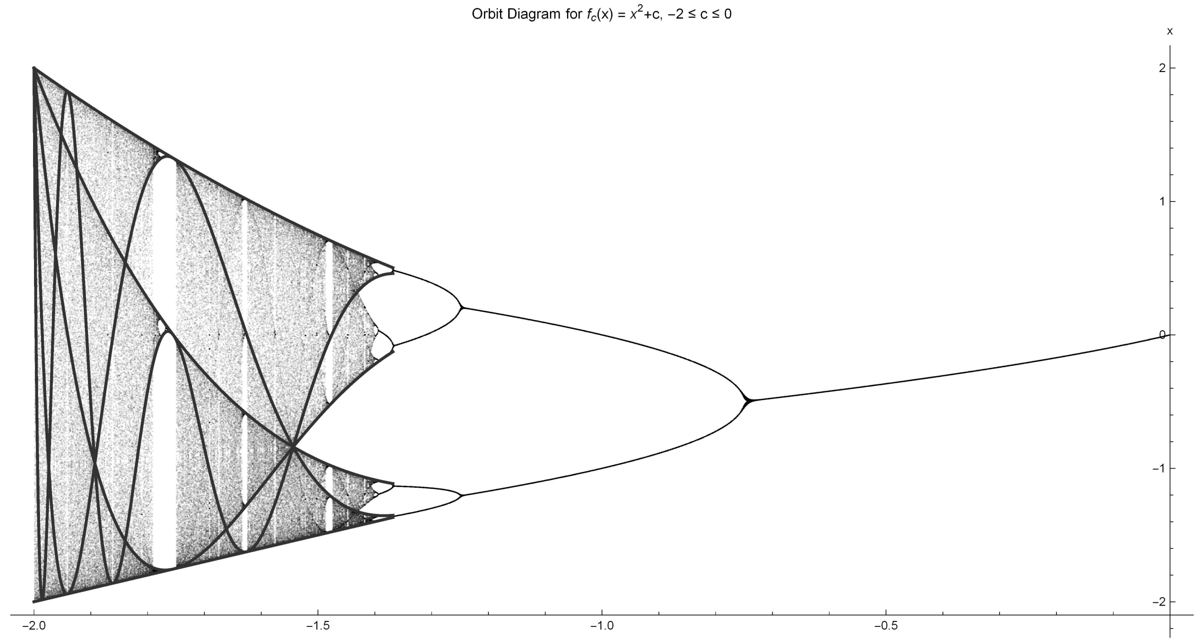
- If we restrict attention near a single point of the apparent 3-cycle, we can see the features more clearly:



- In this picture we can clearly see the period-doubling bifurcations inside this period-3 window.

- <u>Observation 4</u>: On small scales, the orbit diagram has a very similar appearance to the whole orbit diagram. In other words, the orbit diagram exhibits "self-similarity".
  - Notice that the general shape of the diagram inside the period-3 window, for $-1.785 \le c \le -1.75$, looks very much like the original orbit diagram for $-2 \le c \le -0.5$.

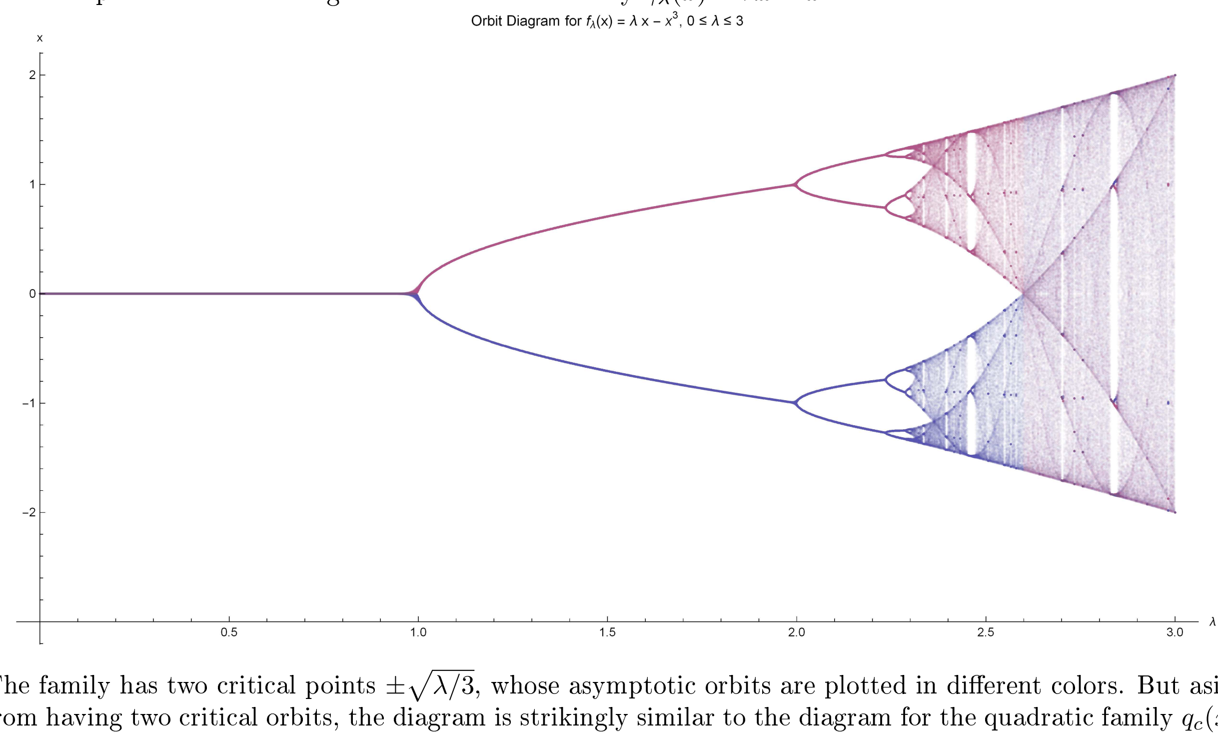
- <u>Observation 5</u>: Inside the "chaotic" regions, there appear to be special (darker) curves running through the picture.
  - Here is a plot of the orbit diagram with some of these curves highlighted:



- It can be shown that these curves are the "critical curves" $x = f_c^n(0)$. (The diagram above includes the critical curves for $n = 1$ through $n = 6$.)

2.3.2 The Cubic Family $\gamma_\lambda(x) = \lambda x - x^3$

- Here is a plot of the orbit diagram[1] for the cubic family $\gamma_\lambda(x) = \lambda x - x^3$:



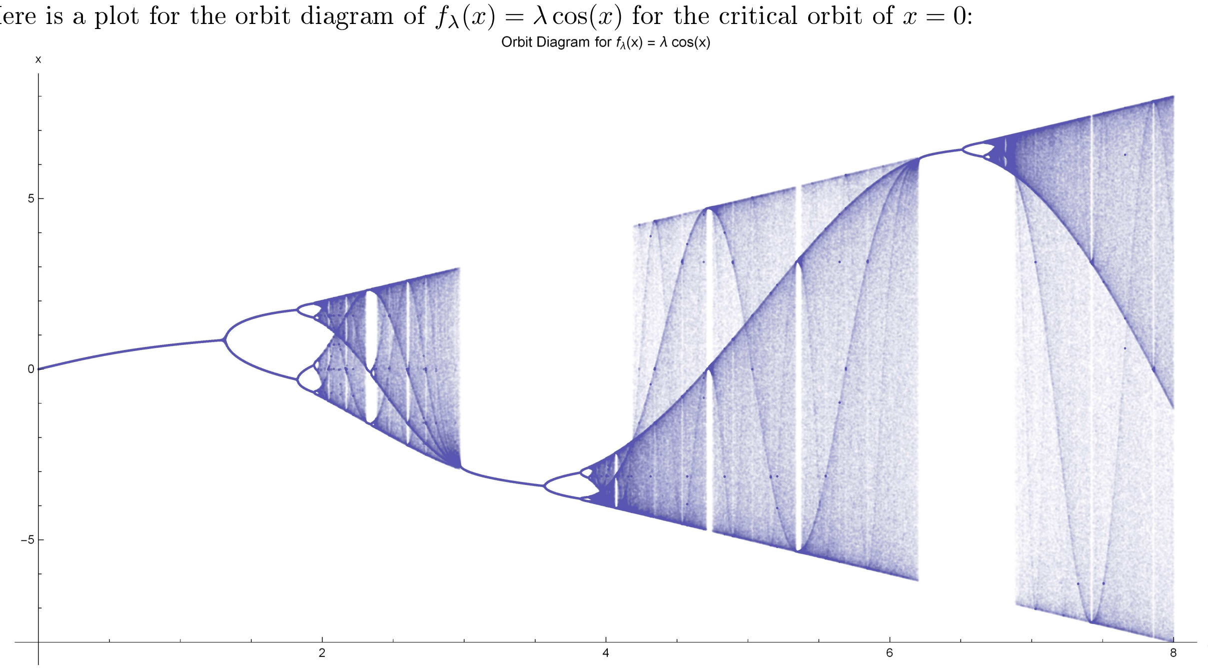
- The family has two critical points $\pm\sqrt{\lambda/3}$, whose asymptotic orbits are plotted in different colors. But aside from having two critical orbits, the diagram is strikingly similar to the diagram for the quadratic family $q_c(x)$: we see a succession of period-doubling bifurcations leading to chaotic behavior, with occasional windows of stability that appear to contain attracting cycles.
  - Notice that the diagram seems to stop at $\lambda = 3$: indeed, if $\lambda = 3$, then the critical orbits are both preperiodic and fall into the 2-cycle $\{2, -2\}$, and if $\lambda > 3$ then the critical orbits will blow up to $\pm\infty$.

[1]The author would like to thank Mark McClure for making Mathematica code available for producing general orbit diagrams. See his post on Mathematica Stack Exchange for more: `http://mathematica.stackexchange.com/questions/13277/mathematica-code-for-bifurcation-diagram`

- For $0 < \lambda < 1$ it is easy to see that the orbits both converge to the attracting fixed point at $x = 0$.
- For λ roughly in the interval $1 < \lambda < 2.60$, the two critical orbits seem to split apart, then transition back to being completely mixed. Indeed, we can actually prove this:
  - For $\lambda > 1$, observe that for $x > 0$, $\gamma_\lambda(x)$ increases from a minimum of 0 to a local maximum value of $2(\lambda/3)^{3/2}$ at $x = \sqrt{\lambda/3}$, and then decreases until it reaches zero at $x = \sqrt{\lambda}$. Therefore, whenever it is true that $2(\lambda/3)^{3/2} \le \sqrt{\lambda}$, we see that γ_λ maps the interval $[0, 2(\lambda/3)^{3/2}]$ into itself.
  - Equality $2(\lambda/3)^{3/2} = \sqrt{\lambda}$ occurs for $\lambda = \dfrac{3\sqrt{3}}{2} \approx 2.5981$. So, we can conclude that for $\lambda \le \dfrac{3\sqrt{3}}{2}$, the positive critical point $\sqrt{\lambda/3}$, which lies in the interval $[0, 2(\lambda/3)^{3/2}]$, will have its entire orbit confined to this interval.
  - By symmetry, the other critical point $-\sqrt{\lambda/3}$ will also have its orbit contained to the symmetric interval $[-2(\lambda/3)^{3/2}, 0]$, so in particular, the two orbits will stay separated.
  - In fact, we can say slightly more: if $2 \le \lambda \le \dfrac{3\sqrt{3}}{2}$, then γ_λ will map the interval $[\gamma_\lambda(2(\lambda/3)^{3/2}),\ 2(\lambda/3)^{3/2}]$ into itself, and thus the orbit of $\sqrt{\lambda/3}$ will be confined to this interval.
  - For λ larger than $\dfrac{3\sqrt{3}}{2}$, the critical orbits will suddenly mix completely, because γ_λ will begin mapping the two intervals $[0, 2(\lambda/3)^{3/2}]$ and $[-2(\lambda/3)^{3/2}, 0]$ into one another.
  - Since 0 is a repelling fixed point for these values of λ, a point that crosses from one interval to the other that lands near zero will be sent far away with the next iteration: this partly explains the "mixing".

2.3.3 The Cosine Family $f_\lambda(x) = \lambda\cos(x)$

- Here is a plot for the orbit diagram of $f_\lambda(x) = \lambda\cos(x)$ for the critical orbit of $x = 0$:

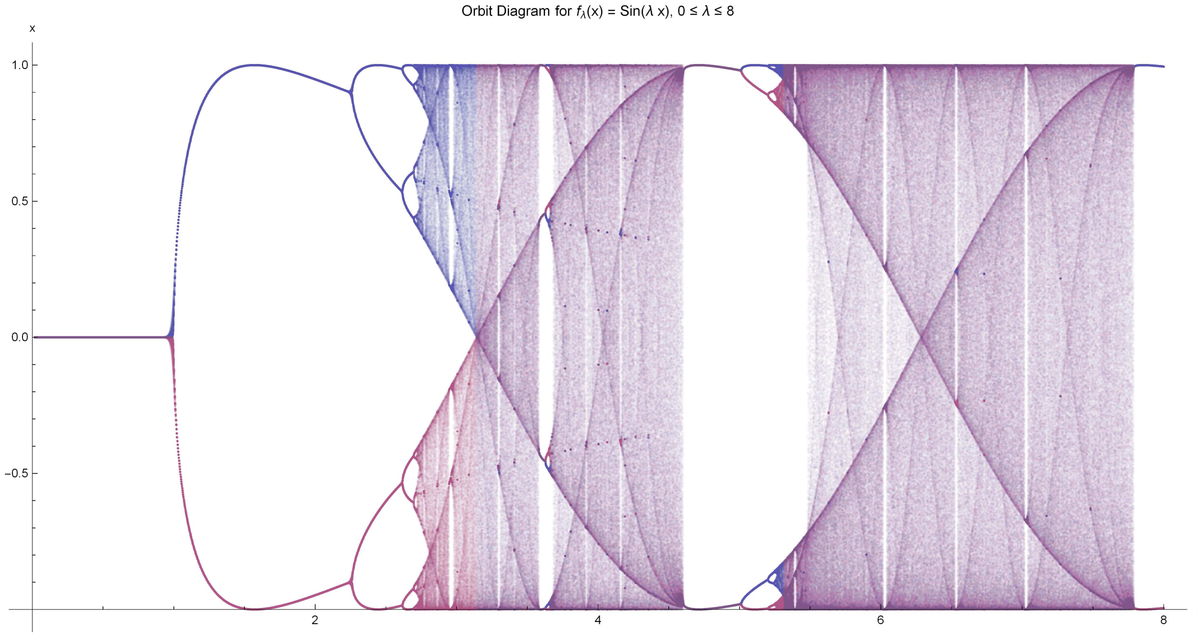


- We note first that, since $f'_\lambda(x) = -\lambda\sin(x)$, the critical points are $x = \pi k$ for integers k. However, although there are infinitely many critical points, all the critical orbits yield the same values after two iterations of f_λ: one easily computes that $f_\lambda^2(\pi k) = \lambda\cos(\lambda)$.
- Also notice that the critical orbit is confined to the interval $[-\lambda, \lambda]$, since this is the range of f_λ. The lines $x = \pm\lambda$ are quite visible in the graph as the upper and lower boundaries of the chaotic regions.

- As with the other orbit diagrams, there is an initial fixed point that undergoes a series of period-doubling bifurcations leading to chaotic behavior. After a chaotic region (with various windows of stability), an attracting fixed point appears at $\lambda \approx 2.97$. At $\lambda \approx 3.57$ there is another series of period-doubling bifurcations which once again leads into a chaotic region, interrupted by windows of stability. An attracting fixed point then reappears at $\lambda \approx 6.20$.
  - We can in fact prove that these attracting fixed points exist: suppose x is a fixed point of f_λ, so that $\lambda\cos(x) = x$: then $f'_\lambda(x) = -\lambda\sin(x) = -x\tan(x)$.
  - A quick analysis using Newton's method indicates that $f'_\lambda(x) = -x\tan(x)$ is between -1 and 1 roughly for $-3.4256 \le x \le -2.7984$, which (from the expression $\lambda = x/\cos(x)$) corresponds to the parameter interval $2.9717 \le \lambda \le 3.5686$.
  - Similarly, $f'_\lambda(x) = -x\tan(x)$ is also between -1 and 1 roughly for $6.1213 \le x \le 6.4373$, which corresponds to the parameter interval $6.2024 \le \lambda \le 6.5145$: so on this interval, f_λ also has an attracting fixed point.
  - By the implicit function theorem we conclude that, on these two parameter intervals, f_λ has an attracting fixed point. On the endpoints of the parameter interval the fixed point becomes neutral and then repelling, which helps to account for the drastic change in the orbit structures.
- We can also see a different kind of change in the orbit structure that occurs at $\lambda \approx 4.19$. Here, it appears that f_λ is chaotic, but transitions from having the orbit be restricted to a small interval to moving through a much larger one.
  - Ultimately, what happens is that for λ less than the transitional value, the critical orbit falls into a small interval that is mapped into itself by f_λ, whereas for λ exceeding the transitional value, the interval is no longer mapped inside itself.
  - More explicitly: by the implicit function theorem, for λ approximately equal to 4.2, there is a fixed point $p(\lambda)$ of f_λ whose value is approximately -2.1. On the interval $[-\lambda, p(\lambda)]$, f_λ decreases from the value $f_\lambda(-\lambda) = \lambda\cos(\lambda)$ to a local minimum $f_\lambda(-\pi) = -\lambda$, and then increases to the value $f_\lambda(p(\lambda)) = p(\lambda)$.
  - This interval is therefore mapped into itself by f provided that $\lambda\cos(\lambda) \le p(\lambda)$: in other words, when $\lambda\cos(\lambda) \le x$, where $\lambda\cos(x) = x$ and x is approximately -2.1. Equality will occur when $\cos(x) = \cos(\lambda)$, which will be the case here when $x = \lambda - 2\pi$ (since x is about -2.1 and λ is about 4.2). Using Newton's method and some calculus, we can see that the inequality $\lambda\cos(\lambda) \le \lambda - 2\pi$ holds when $\lambda \le 4.1888$ and fails for larger λ.
  - Furthermore, the fixed point x is approximately $\lambda - 2\pi \approx -2.0944$, and $f'_\lambda(-2.0944) \approx 3.6376$. Thus, once λ exceeds this transitional value, the interval $[-\lambda, p(\lambda)]$ will be mapped outside itself, and points near $p(\lambda)$ will be pushed far away (since the point is repelling). This accounts for the sudden "enlargement" of the asymptotic orbit.

2.3.4 The Sine Family $f_\lambda(x) = \sin(\lambda x)$

- Here is a plot for the orbit diagram of $f_\lambda(x) = \sin(\lambda x)$ for its two critical orbits:



- Note that, as in the cosine family, each f_λ has an infinite number of critical points; namely, $\dfrac{\pi(1+2k)}{2\lambda}$ for integers k. But as before, we are only interested in asymptotic behavior, so since the value of f_λ at each critical point is ± 1, we only need to plot the orbits of ± 1 under f_λ.
- Like with the cubic family, we see that the two critical orbits align for a brief time on a single attracting fixed point, then separate into distinct orbits until approximately $\lambda = \pi$, and then eventually mix uniformly. (We could prove these statements using arguments similar to those we gave for the cubic family.)
- With this family we also see an interesting reoccurrence of stable behavior once λ is approximately $3\pi/2$.
  - Here, it is easy to verify the existence of an attracting 2-cycle $\{-1, 1\}$ at $\lambda = 3\pi/2$, as $f'_\lambda(\pm 1) = 0$. As λ increases, this 2-cycle undergoes a series of period-doubling bifurcations that lead back into chaos.
  - The first period-doubling splits the two critical orbits apart into different cycles, which then separately undergo period-doublings until they become chaotic, and then eventually become mixed together.
  - Similar reappearances of stability will also occur at $\lambda \approx 7\pi/2$, $11\pi/2$, $15\pi/2$, and so forth.

Well, you're at the end of my handout. Hope it was helpful.