E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Summer-2 2023 ~ Homework 7, due Fri Aug 18th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the fixed points of each complex function and classify them as attracting, repelling, or neutral:

   (a) $f(z) = z + z^2(z - i)(z + 1)$.

   (b) $f(z) = z^2 - 3z + 5$.

   (c) $f(z) = iz^2 + z + i/4$.

   (d) $f(z) = 1 + \dfrac{4i}{z + 2 - 3i}$.

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2. For each function $f(z)$ and each value z_0, show that z_0 is a periodic point for f and classify the associated cycle as attracting, repelling, or neutral:

   (a) $f(z) = 1 - i + iz$ with $z_0 = 2$.

   (b) $f(z) = z^2 + i$ with $z_0 = -i$.

   (c) $f(z) = 1 - \dfrac{3}{2}z^2 - \dfrac{1}{2}z^3$ with $z_0 = 0$.

   (d) $f(z) = 3z + 4/z$ with $z_0 = i$.

   (e) $f(z) = z^2$ with $z_0 = \cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9} = e^{2\pi i/9}$.

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3. For the following quadratic functions, (i) plot the Julia set / filled Julia set, (ii) use the picture to identify whether the Julia set for that function is connected or disconnected, and then (iii) justify your answer using the fundamental dichotomy.

   (a) $f_1(z) = z^2 - 0.4 - 0.1i$.

   (b) $f_2(z) = z^2 + 0.2 - i$.

   (c) $f_3(z) = z^2 + i$.

   (d) $f_4(z) = z^2 - 0.53 + 0.6i$.

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4. For the following non-quadratic functions, (i) plot the Julia set and filled Julia set and (ii) use the picture to identify whether the Julia set for that function seems to be connected or disconnected.

   (a) $f_5(z) = z^3 + 0.4z - i$.

   (b) $f_6(z) = \dfrac{z^3 - 1}{z + i}$.

   (c) $N(z) = z - \dfrac{z^3 - 1}{3z^2}$.

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5. Each of the following values of c lies in a periodic bulb of the Mandelbrot set. For each value, (i) identify the p/q labeling of that bulb based on its location, (ii) check the result of the "antenna theorem" for the bulb, and (iii) check the result of the "lobe theorem" for the filled Julia set corresponding to that value of c. (Note that it may be difficult to visually judge the "smallest" antenna of the Mandelbrot bulb or the "largest" lobes of the Julia set, and at least one of the results will not agree with the prediction of the corresponding theorem!)

   (a) $c = -0.1 + 0.7i$.

   (b) $c = 0.3 + 0.55i$.

   (c) $c = -0.5 + 0.55i$.

   (d) $c = -0.625 + 0.425i$.

   (e) $c = 0.05 + 0.63i$.

   (f) $c = 0.375 + 0.27i$.

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6. There are many bulbs of the Mandelbrot set that are not attached directly to the main cardioid, but rather to another bulb. Each of these values of c lies in a "secondary bulb" attached to a "primary bulb" of the main cardioid: identify the p/q labeling of the primary bulb to which it is attached, and find the period inside the secondary bulb. Is there any relation between the periods in the secondary bulb and the primary bulb?

   (a) $c = -0.21 + 0.8i$.

   (b) $c = 0.13 - 0.63i$.

   (c) $c = 0.390 + 0.232i$.

   (d) $c = -0.547 - 0.557i$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

7. Consider the holomorphic function $f_k(z) = z^k$, where $k \geq 2$ is an integer.

   (a) Show that, except for $z = 0$, every eventually periodic point of f_k lies on the unit circle $|z| = 1$.

   (b) Show that every periodic cycle for f_k, except for the fixed point $z = 0$, is repelling.

   (c) If $z = e^{2\pi i t}$ where $t \in [0, 1]$, show that z is eventually periodic for f_k if and only if t is a rational number.

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8. The goal of this problem is to explain some symmetries in the Julia sets we have examined.

   (a) Show that the Julia set for any map in the quadratic family $q_c(z) = z^2 + c$ is symmetric about the origin.

   (b) Show that the Julia set for the map $q_c(z) = z^2 + c$ is the reflection through the real axis of the Julia set for the map $q_{\overline{c}}(z) = z^2 + \overline{c}$.

   (c) Show that if c is a real number, then the Julia set for the map $q_c(z) = z^2 + c$ is symmetric about the real and imaginary axes.

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9. Let $p_{d,c}(z) = z^d + c$, where $d \geq 3$ is an integer, and observe that $p_{d,c}$ has a single critical point at $z = 0$. By the theorem on the structure of Julia sets for polynomial maps, we see that the Julia set for the map $p_{d,c} = z^d + c$ is either a totally disconnected Cantor-like set, or consists of a single connected component, according to whether the orbit of 0 escapes to ∞ or not. Define the <u>multibrot set</u> M_d to be the set of points $c \in \mathbb{C}$ for which the Julia set of $p_{d,c}$ is connected: equivalently, M_d is the set of points for which the orbit of 0 under $p_{d,c}$ remains bounded.

   (a) Compare the sets M_3, M_4, M_5, and M_6 to one another and to the Mandelbrot set M. [In Mandel, you can plot these sets using the "Function" menu. Typing "q" will open a menu to change the value of d.]

   (b) Show that if $|z| > \max(|c|, 2^{1/(d-1)})$, then there is a $\lambda > 1$ such that $|p_{d,c}(z)| \geq \lambda |z|$. Conclude that for any such z, the orbit of z under $p_{d,c}$ escapes to ∞. [Hint: Modify the proof of the escape criterion for quadratic maps.]

   (c) Show that every point in the multibrot set M_d lies within or on the circle of radius $2^{1/(d-1)}$ centered at the origin. [Hint: Consider the second iterate of 0 and use part (b).]

   (d) Show that $p_{d,c}(z)$ is conjugate to $p_{d,\omega c}(z)$, where $\omega = e^{2\pi i/(d-1)}$.

   (e) Show that the multibrot set M_d for any $d \geq 3$ is invariant under rotation by an angle of $2\pi/(d-1)$ radians about the origin. [Hint: Use part (d).]
