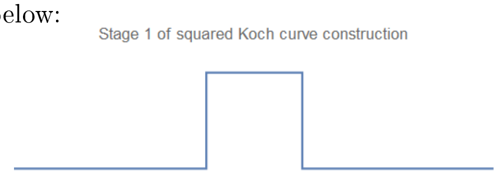
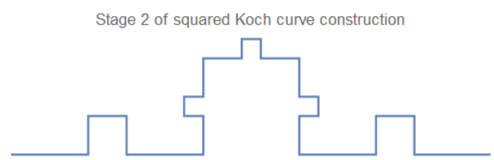
E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Summer-2 2023 ~ Homework 6, due Fri Aug 11th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. The goal of this problem is to study a "squared" version of the Koch curve fractal, constructed as follows: let E_0 be a line segment of length 1. Then, for each $n \geq 1$, define the set E_n to be the set obtained by removing the middle fifth of each segment in E_{n-1} and replacing it with the other three sides of the outwards square sharing those endpoints. The <u>squared Koch curve</u> is the limiting set as $n \to \infty$. The first two iterates are shown below:





   (a) Plot the 3rd, 4th, 5th, and 6th iterates of the construction.
   (b) Compute the total length of the graph of the nth stage of the construction. What happens as $n \to \infty$?
   (c) Compute the total new area created under/inside the graph of the nth stage of the construction. What happens as $n \to \infty$? [Hint: The new area produced in each stage is a constant times the area produced in the previous stage.]
   (d) Find the box-counting dimension of the squared Koch curve, to at least three decimal places.

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2. The binomial coefficients $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are often displayed in an array called Pascal's triangle.

   (a) Describe the result obtained by (re)drawing the array with a black dot in place of each binomial coefficient that is odd, and with a white dot in place of each binomial coefficient that is even. Try explaining the result using the recurrence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
   (b) What happens if instead you plot the points where the binomial coefficient is congruent to 0 modulo 3? 1 modulo 3? 2 modulo 3? Based on the picture, what is the box-counting dimension of the set where the binomial coefficients are 1 or 2 modulo 3? What do you think will happen with other moduli?

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3. If you run the command `Nest[Subsuperscript[#, #, #] &, x, 6]` in Mathematica, a fractal will appear. Which fractal, and why?

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4. For each iterated function system inside the unit square $[0,1] \times [0,1]$, (i) use the chaos game to plot the invariant set, and (ii) compute the box-counting dimension for the invariant set to at least 3 decimal places.

   (a) $\{f_1, f_2, f_3, f_4, f_5\}$, where $f_1(x,y) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x,y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$, $f_3(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_4(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, $f_5(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.
   (b) $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$, where $f_1(x) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$, $f_3(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$, $f_4(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_5(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, $f_6(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_7(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.
   (c) $\{f_1, f_2, f_3\}$, where $f_1(x) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$, $f_3(x) = (\frac{2}{3}x + \frac{1}{3}, \frac{2}{3}y + \frac{1}{3})$.
   (d) $\{f_1, f_2, f_3\}$, where $f_1(x) = (\frac{1}{2}x, \frac{1}{2}y)$, $f_2(x) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y)$, $f_3(x) = (1 - \frac{1}{2}y, \frac{1}{2}x + \frac{1}{2})$.
   (e) $\{f_1, f_2, f_3, f_4\}$, where $f_1(x) = (\frac{1}{3} - \frac{1}{3}y, \frac{1}{3}x)$, $f_2(x) = (1 - \frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$, $f_3(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{2}{3})$, $f_4(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.

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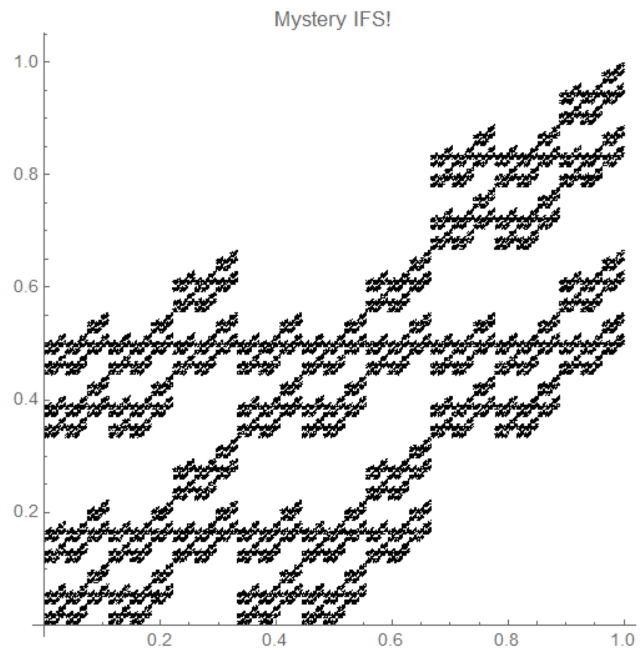
5. Use the chaos game to plot 10 000, 30 000, 100 000, and 300 000 points, each with point size 0.001, for each of the iterated function systems in problem 4, and determine the smallest value among those four that produces a "good" picture of the fractal. For the four systems you analyzed, is there any relation between the number of points needed for a good picture and the box-counting dimension of the set?

6. Find an iterated function system inside the square $[0,1] \times [0,1]$ whose invariant set is as pictured. [Hint: Break the set into a disjoint union of smaller copies of itself, and then identify the similarities that map the original set onto each smaller copy.]



Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

7. The Vicsek box fractal is constructed as follows: begin with a unit square. Then divide it into nine equal subsquares, and then remove the four squares that touch a midpoint of one of the sides. Now apply this procedure to each of the five smaller squares, thus creating 25 smaller squares, and continue iterating. The Vicsek box fractal is the limiting set from this procedure.

   (a) Draw the first two iterations of the Vicsek box fractal.

   (b) Find the total area and perimeter of the nth iterate and determine what happens to each as $n \to \infty$.

   (c) Show that the topological dimension of the Vicsek box fractal is 1. [Hint: Show that the box fractal contains a line, and also that, for any of the individual squares in the nth iterate, there is a circle that passes through its vertices but no other points in the set.]

   (d) Find the box-counting dimension of the Vicsek box fractal.

8. The goal of this problem is to analyze a few variations on the center-2/5 Cantor set.

   Define the left-center-2/5 Cantor set as follows: we begin with $[0,1]$, and then at each stage, we divide each remaining interval into five equal pieces and remove the open second and third pieces. Thus, the first iterate consists of the two intervals $[0,1/5] \cup [3/5,1]$.

   (a) Show that the topological dimension of the left-center-2/5 Cantor set is 0. [Hint: Show that between any two points in the set, there is at least one point not in the set.]

   (b) Find the box-counting dimension of the left-center-2/5 Cantor set, to at least 3 decimal places.

   Now define the alternating-2/5 Cantor set as follows: we begin with $[0,1]$, and then at each stage, we divide each remaining interval into five equal pieces and remove the open second and fourth pieces. Thus, the first iterate consists of the three intervals $[0,1/5] \cup [2/5,3/5] \cup [4/5,1]$.

   (c) Show that the topological dimension of the alternating-2/5 Cantor set is 0.

   (d) Find the box-counting dimension of the alternating-2/5 Cantor set, to at least 3 decimal places.