E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Summer-2 2023 ~ Homework 5, due Fri Aug 4th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For the following dynamical systems (X, f), does f have sensitive dependence on all of X? If not, does it have sensitive dependence at any individual points in X? (Give some justification.)
   (a) $X = [0, 1]$, $f(x) = x^3$.
   (b) $X = \mathbb{R}$, $f(x) = \cos(x)$.
   (c) $X = \Sigma_2$, $f(d_0 d_1 d_2 d_3 d_4 d_5 \cdots) = (d_2 d_3 d_4 d_5 \cdots)$.

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2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Answer the following:
   (a) If f has a point of exact period 11, does it necessarily have a point of exact period 32? 26? 17? 9? 4?
   (b) If f has a point of exact period 22, does it necessarily have a point of exact period 32? 26? 17? 9? 4?
   (c) If f has a point of exact period 16, does it necessarily have a point of exact period 32? 26? 17? 9? 4?

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3. Let $g : [0, 2] \to [0, 2]$ be defined via $g(x) = \begin{cases} x^2 - 3x + 2 & \text{for } 0 \le x \le 1 \\ x - 1 & \text{for } 1 < x \le 2 \end{cases}$. Show that g has a point of exact period n for each $n \ge 1$.

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4. Find exact expressions for the three 4-cycles of $q_{-2}(x) = x^2 - 2$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Recall the definition of the tripling map $T(x) = \begin{cases} 3x & \text{for } 0 \le x < 1/3 \\ 3x - 1 & \text{for } 1/3 \le x < 2/3 \\ 3x - 2 & \text{for } 2/3 \le x < 1 \end{cases}$ on the interval $[0, 1)$. Equivalently, $T(x) = 3x$ modulo 1. The goal of this problem is to prove that T is chaotic.
   (a) Suppose $\alpha = 0.d_1 d_2 d_3 d_4 \ldots$ in base 3, where, if α has a terminating expansion, we use that expansion instead of the non-terminating one. Show that $T(\alpha) = 0.d_2 d_3 d_4 \ldots$.
   (b) Show that the periodic points for T in $[0, 1)$ are precisely the points with a periodic base-3 decimal expansion. Conclude that the set of periodic points for T is dense in $[0, 1)$.
   (c) Let $\gamma = 0.\underbrace{0\,1\,2}_{\text{length 1}}\ \underbrace{00\,01\,02\,10 \cdots 22}_{\text{length 2}} \cdots$ be the real number obtained by listing all length-1 sequences in base 3, then all length-2 sequences, then all length-3 sequences, and so forth. Show that the orbit of γ is dense in $[0, 1)$, and conclude T is transitive on $[0, 1)$.
   (d) Suppose x and y are in $[0, 1)$ with $y > x$. If both x and y lie in the same interval $[0, 1/3)$, $[1/3, 2/3)$, or $[2/3, 1)$, show that $|T(y) - T(x)| = 3\,|y - x|$. If they lie in different intervals, show that at least one of $|y - x|$ and $|T(y) - T(x)|$ must be $\ge 1/4$.
   (e) Suppose x and y are in $[0, 1)$ and that $y \ne x$. Show that there is some $k \ge 0$ for which $\left|T^k(y) - T^k(x)\right| \ge 1/4$. Conclude that T has sensitive dependence on $[0, 1)$.
   (f) Show that T is chaotic on $[0, 1)$.

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6. Let the tent map be $T(x) = \begin{cases} 2x & \text{for } 0 \le x < 1/2 \\ 2 - 2x & \text{for } 1/2 \le x \le 1 \end{cases}$, and then, for $0 \le h \le 1$, define the truncated tent map to be $T_h(x) = \min(h, T(x))$. The goal of this problem is to explore how these maps can be used to prove the converse of Sarkovskii's theorem.

   (a) Find the 2-cycles, 3-cycles, 4-cycles, and 5-cycles for the map T. (There are 1, 2, 3, and 6 respectively.)

   (b) Suppose $0 < h \le 1$ and $m \ge 1$. Show that any m-cycle $\{x_1, x_2, \cdots, x_m\}$ for T_h is also an m-cycle for T, except possibly if some $x_i = h$. [Hint: $T(x) = T_h(x)$ whenever $T(x) \le h$.]

   (c) Suppose that $\{x_1, x_2, \cdots, x_m\}$ is an m-cycle for T, and $\alpha = \max(x_1, x_2, \ldots, x_m)$. Show that $\{x_1, x_2, \cdots, x_m\}$ is an m-cycle for T_h for all $\alpha \le h \le 1$, but is not an m-cycle for T_h when $h < \alpha$. [Hint: $T_h(x) = T_\alpha(x)$ whenever $x \le \alpha$; for the second part, consider the range of T_h.]

   (d) Show that the map $T_{4/5}$ has cycles of lengths 2 and 1 but no others.

   (e) Show that the map $T_{28/33}$ has cycles of every length except 3.

   (f) Show that the map $T_{106/127}$ has cycles of every length except 3 and 5.

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7. Suppose $f : I \to I$ is a continuous function on a closed interval $I \subseteq \mathbb{R}$.

   (a) Suppose that the set of fixed points of f is dense in I. Prove that f must be the identity function (i.e., that $f(x) = x$ for all $x \in I$). [Hint: If $\lim_{n\to\infty} a_n = a$ is any convergent sequence, then $\lim_{n\to\infty} f(a_n) = f(a)$.]

   (b) Suppose that the set of periodic points of period $\le n$ for f is dense in I. Prove that some iterate of f must be the identity function. [Hint: Consider $g = f^{n!}$.]

   (c) [Optional] Suppose that some iterate of f is the identity function. Show that f cannot have sensitive dependence. [Hint: If f^n is the identity, then given $x \in I$, show that there exists a positive constant ϵ_k such that $|y - x| < \epsilon_k$ implies $\left|f^k(y) - f^k(x)\right| < \beta/2$. Then let $\epsilon = \min(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1})$.]

   (d) Suppose that f is chaotic on I. Show that f must have periodic points of arbitrarily large exact period.

   (e) Suppose that f is chaotic on I. Show that f must have a point of exact order 2^d for every positive integer d.
