E. Dummit's Math 3543 ∼ Dynamics, Chaos, and Fractals, Summer-2 2023 ∼ Homework 4, due Fri Jul 28th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let Σ_2 be the binary sequence space with its standard metric $d(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$.

   (a) Find $d(\mathbf{x}, \mathbf{y})$, $d(\mathbf{x}, \mathbf{z})$, and $d(\mathbf{y}, \mathbf{z})$, where $\mathbf{x} = (0\overline{10})$, $\mathbf{y} = (01\overline{1})$, and $\mathbf{z} = (1\overline{10})$.

   (b) Is there a sequence $\mathbf{y}$ such that $d(\mathbf{x}, \mathbf{y}) = \frac{1}{3}$, if $\mathbf{x} = (1\overline{0})$? Explain why or why not.

   (c) Is there a sequence $\mathbf{y}$ such that $d(\mathbf{x}, \mathbf{y}) = \frac{1}{3}$ and $d(\mathbf{y}, \mathbf{z}) = \frac{1}{3}$, if $\mathbf{x} = (1\overline{0})$ and $\mathbf{z} = (\overline{0})$? Explain why or why not.

   (d) Describe the points that lie in the open ball of radius 1 centered at $\mathbf{x} = (\overline{0})$: in other words, the points $\mathbf{y}$ with $d(\mathbf{x}, \mathbf{y}) < 1$.

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2. Consider the shift map σ on the binary sequence space Σ_2.

   (a) How many cycles of length 4, 5, and 6 are there?

   (b) Find explicitly all of the 4-cycles and all of the 5-cycles.

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3. Show that the given dynamical systems either are conjugate (by finding an explicit homeomorphism between them; you need not show it is actually a homeomorphism) or are not conjugate (by demonstrating a difference in their orbit structures):

   (a) $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ where $f(x) = x^3 + 6x^2 + 12x + 6$ and $g(x) = x^3$.

   (b) $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ where $f(x) = x^2$ and $g(x) = x^3$.

   (c) $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ where $f(x) = 3x$ and $g(x) = x^3$.

   (d) $(\mathbb{R}, f)$ and (Y, g) where $f(x) = 3x$, $g(x) = x^3$, and $Y = (0, \infty)$. [Hint: Exponentials.]

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4. For the following pairs (X, S), is S a dense subset of X or not? (Give brief justification for your answers.)

   (a) $X = \mathbb{R}$, $S =$the numbers of the form $\frac{k}{2^n}$ for k, n integers.

   (b) $X = [0, 1]$, $S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$.

   (c) $X = [0, 1]$, $S =$the Cantor ternary set.

   (d) $X = [0, 1]$, $S =$all points in X not in the Cantor ternary set.

   (e) $X = \Sigma_2$, $S =$the sequences of the form $(d_0 d_1 d_2 \cdots)$ having only finitely many 1s.

   (f) $X = \Sigma_2$, $S =$the orbit under σ of $\beta = (101001000100001 \cdots)$ whose expansion continues the pattern of a single 1 followed by one more 0 than the previous cluster of 0s.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. The goal of this problem is to investigate some properties of the Cantor ternary set $\Gamma = \bigcap_{n=0}^{\infty} C_n$, where (recall) $C_0 = [0, 1]$ and C_{n+1} is obtained by removing the open middle third of each interval in C_n. Equivalently, Γ consists of the points in $[0, 1]$ that have a base-3 decimal expansion containing no 1s.

   (a) Find the total length of all the intervals in C_n, and show that it goes to zero exponentially fast as $n \to \infty$.

   (b) Show that every point in Γ is equal to a limit of a sequence of other points of Γ. [Hint: Use base-3 expansions, but be careful with terminating expansions!]

   (c) Show that Γ contains no nontrivial intervals (i.e., no intervals containing more than a single point). Conclude that if $x < y$ are any two points in Γ, then there exists a z with $x < z < y$ such that z is not in Γ.

   (d) Show that $\Gamma + \Gamma = [0, 2]$: in other words, show that every real number in the interval $[0, 2]$ can be written as the sum of two (not necessarily different) elements of Γ. [Hint: Consider the base-3 decimal expansions of elements in the set $\frac{1}{2}\Gamma = \{\frac{1}{2}x : x \in \Gamma\}$.]

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6. Consider the "tent map" $T(x) = \begin{cases} 3x & \text{if } x \le 1/2 \\ 3 - 3x & \text{if } 1/2 < x \end{cases}$. (Its name comes from the shape of its graph.)

   (a) Show that if x is outside $[0, 1]$ then $T^n(x) \to -\infty$ as $n \to \infty$.

   (b) Show that the set of points x such that $T(x) \in [0, 1]$ is the union of two closed intervals, and identify these intervals.

   (c) Show that the set of points x such that $T^2(x) \in [0, 1]$ is the union of four closed intervals, and identify these intervals.

   (d) Identify the set of points x such that $T^n(x) \in [0, 1]$ for every $n \ge 1$, and then prove it.

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7. Let $q(x) = x^2 - 6$.

   (a) Show that q has exactly 30 real-valued 8-cycles.

   (b) Show that all of the 8-cycles lie in the interval $[-3, 3]$.

   (c) Using the function NSolve, ask Mathematica to numerically compute the roots of $q^8(x) - x = 0$. Are the results correct? Then ask Mathematica to find the roots of $\dfrac{q^8(x) - x}{q^4(x) - x} = 0$: are these results correct? If you have access to another computer algebra system, try asking it the same questions: does it return correct results?
