E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Summer-2 2023 ~ Homework 3, due Fri Jul 21st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Compute the Schwarzian derivative $Sf(x)$ for each function $f(x)$, and decide whether $Sf(x) < 0$ for all x:
   (a) $f_1(x) = \dfrac{x}{x+1}$.
   (b) $f_2(x) = x^a$ for a a constant. (The answer will depend on a.)
   (c) $f_3(x) = \tan(x)$.
   (d) $f_4(x) = x^4 - 3x + \pi$.

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2. Each of the given one-parameter families $f_\lambda(x)$ has a bifurcation at the given value of λ_0. Plot the bifurcation diagram for the family and use it to identify the type of bifurcation there, and then show algebraically that the claimed bifurcation does occur:
   (a) $f_\lambda(x) = \lambda \sin(x)$, $\lambda_0 = -1$.
   (b) $f_\lambda(x) = \lambda e^x - 2$, $\lambda_0 = e$.
   (c) $f_\lambda(x) = e^{\lambda x}$, $\lambda_0 = -e$.
   (d) $f_\lambda(x) = \lambda x^2 - x^3$, $\lambda_0 = 2$.
   (e) $f_\lambda(x) = \lambda x^2 - x^3$, $\lambda_0 = 4/\sqrt{3}$.

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3. Let $f_\lambda(x) = \lambda x - x^3$.
   (a) Plot the bifurcation diagram for this family. (Include fixed points and 2-cycles.)
   (b) Identify the three pairs (λ_0, x_0) where period-doubling bifurcations occur, and then show algebraically that period-doubling bifurcations do occur there.
   (c) There is another bifurcation at $\lambda_0 = 1$ that is called a "pitchfork" bifurcation. Explain why it is not a saddle-node bifurcation, according to our definition.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose f is continuously differentiable and has finitely many zeroes $r_1 < r_2 < \cdots < r_n$ each having finite multiplicity ≥ 1.
   (a) Show that the Newton iterating function $N(x)$ is undefined somewhere in the interval (r_i, r_{i+1}) for each i with $1 \leq i \leq n-1$. [Hint: Use the mean value theorem.]
   (b) If $i \neq 1, n$, show that the immediate attracting basin for r_i as a fixed point of N must have the form (a, b) where $\{a, b\}$ is a 2-cycle for N. [Hint: Explain why the immediate basin does not extend to $\pm\infty$, and then use this to show that N cannot be undefined at either endpoint of the basin.]
   (c) For $f(x) = x(x-1)(x-4)$, find the immediate attracting basin for the fixed point $r_2 = 1$ of the Newton iterating function for f. (Give your answer to five decimal places.)

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5. If we zoom in on the orbit diagram for $q_c(x) = x^2 + c$, there appears to be an attracting 3-cycle when $c = -1.76$.

   (a) Using the asymptotic orbit of the critical point $x = 0$, compute, to 10 decimal places, the apparent points on this 3-cycle.

   (b) For $f(x) = x^2 - 1.76$, verify that if $I_1 = (0.0236, 0.0241)$, $I_2 = (-1.75945, -1.75941)$, and $I_3 = (1.33552, 1.33567)$ then $f(I_1) \subseteq I_2$, $f(I_2) \subseteq I_3$, and $f(I_3) \subseteq I_1$.

   (c) Show that there is indeed an attracting 3-cycle for $q_c(x)$ when $c = -1.76$.

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6. Consider the one-parameter family $f_\lambda(x) = \lambda \cos(x)$ for $\lambda > 0$.

   (a) Show that there is a unique asymptotic critical orbit, and that all of its points lie in $[-\lambda, \lambda]$.

   (b) Plot the orbit diagram for $0 \leq \lambda \leq 8$.

   (c) Describe the change in the orbit structure that occurs for $\lambda \approx 2.97$. Can you explain it?

   (d) Describe the change in the orbit structure that occurs for$\lambda \approx 6.20$. Can you explain it?

   (e) Describe the change in the orbit structure that occurs for$\lambda \approx 4.19$. Can you explain it?

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7. The goal of this problem is to illustrate how the Schwarzian derivative was originally used in complex analysis to characterize the fractional linear transformations $f(x) = \dfrac{ax+b}{cx+d}$, where a, b, c, d are constants.

   (a) Show that $S(1/x) = 0$ and $S(cx+d) = 0$ for any c, d.

   (b) Suppose that $Sf = 0$ and $Sg = 0$. Show that $S(f \circ g) = 0$ also. [Hint: Schwarzian chain rule.]

   (c) Show that $S(\dfrac{1}{cx+d}) = 0$ and then that $S(\dfrac{ax+b}{cx+d}) = S(\dfrac{(bc-ad)/c}{cx+d} + \dfrac{a}{c}) = 0$ for any a, b, c, d.

   Part (c) shows that the Schwarzian derivative of any fractional linear transformation is zero. Our goal now is to show the converse: that a function with Schwarzian derivative zero must be a fractional linear transformation.

   (d) [Optional] Suppose f is such that $Sf(x) = 0$ for all x. Show that $\ln(f'') = \dfrac{3}{2}\ln(f') + C$ for some constant C. [Hint: Integrate $\dfrac{f''}{f'''} = \dfrac{3}{2}\dfrac{f''}{f'}$.]

   (e) [Optional] Suppose g is such that $\ln(g') = \dfrac{3}{2}\ln(g) + C$ for some constant C. Show that $g(x) = (cx+d)^{-2}$ for some c and d. [Hint: Integrate $g^{-3/2}g' = e^C$.]

   (f) [Optional] Suppose $Sf(x) = 0$ for all x. Show that $f(x) = \dfrac{ax+b}{cx+d}$ for some a, b, c, d.
