

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Each of these functions has a neutral fixed point. Find it (you do not need to show it is unique), and then determine whether it is weakly attracting or weakly repelling for orbits on each side:

(a)  $m(x) = x + x^5$ .

(b)  $a(x) = x - x^4 + x^7$ .

(c)  $t(x) = \sin(x)$ .

(d)  $h(x) = e^{x/e}$ .

(e)  $y(x) = \ln(1 - x)$ .

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2. Find the immediate attracting basin of the attracting (or weakly attracting) fixed point  $x_0 = 0$  for each of these functions:

(a)  $f(x) = x - x^5$ .

(b)  $u(x) = x^3 + \frac{3}{4}x$ .

(c)  $n(x) = \frac{2x^2}{3x - 1}$ .

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3. Let  $p_c(x) = -x + x^2 + cx^3$ , where  $c$  is a constant, and notice that 0 is a neutral fixed point. Determine (in terms of  $c$ ) when 0 is weakly attracting and when it is weakly repelling.
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4. Let  $p(x) = x^3 - ax$  for a parameter  $a > 1$ .

(a) Find the three fixed points of  $p$  and classify them as attracting, repelling, or neutral (in terms of  $a$ ).

(b) Find a 2-cycle for  $p$  of the form  $\{x_0, -x_0\}$  and classify it as attracting, repelling, or neutral (in terms of  $a$ ).

(c) For  $a = 5/2$ , it turns out that  $p$  has two other 2-cycles in addition to the one you found in part (b). Compute them explicitly, and classify them as attracting, repelling, or neutral.

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5. Let  $f(x) = x^3 - 2x - 2$ .

(a) Show that this polynomial has exactly one real root  $r$  and that it lies in the interval  $(1, 2)$ .

(b) Find the Newton iterating function for  $f$  and then use Newton's method to calculate the root  $r$ , accurate to 10 decimal places.

(c) Compare your result in part (b) to the numerical value of  $\frac{1}{3} \left[ \sqrt[3]{27 + 3\sqrt{57}} + \sqrt[3]{27 - 3\sqrt{57}} \right]$ .

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6. Let  $p(x) = x^5(x - 1)^2(x - 2)$ .

(a) Find the Newton iterating function  $N(x)$  for  $p$ , along with the fixed points of  $N$ .

(b) Give a table of the first 10 iterates of the Newton iterating function applied to the starting values 0.1, 1.1, and 2.1. Which root has the fastest convergence? Which root has the slowest convergence? What does Newton's fixed point theorem predict?

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear explanations.

7. Suppose  $\{x_0, x_1, x_2\}$  is a neutral 3-cycle for the function  $f$  and  $f'(x_i) = 1$  for each  $i = 0, 1, 2$ .

- (a) If  $g = f^3$ , show that  $g''(x_0) = g''(x_1) = g''(x_2) = f''(x_0) + f''(x_1) + f''(x_2)$ . [Hint: Use the product and chain rules to compute  $g''(x)$ , then set  $x = x_0$ .]
- (b) Let  $p(x) = 1 + x - 3x^2 - \frac{15}{4}x^3 + \frac{3}{2}x^4 + \frac{9}{4}x^5$ . Show that 0 lies on a neutral 3-cycle for  $p$ , and classify the behavior of  $p^3$  near 0 as weakly attracting or repelling on each side of 0.

8. Consider the two functions

$$s_1(x) = -x + x^2 - x^3 + \frac{3}{2}x^4 - \frac{5}{2}x^5 + \frac{19}{8}x^6 - \frac{139}{80}x^8 + \frac{653}{1000}x^{10}$$

$$s_2(x) = -x + x^2 - x^3 + \frac{3}{2}x^4 - \frac{5}{2}x^5 + \frac{19}{8}x^6 - \frac{139}{80}x^8 + \frac{652}{1000}x^{10}$$

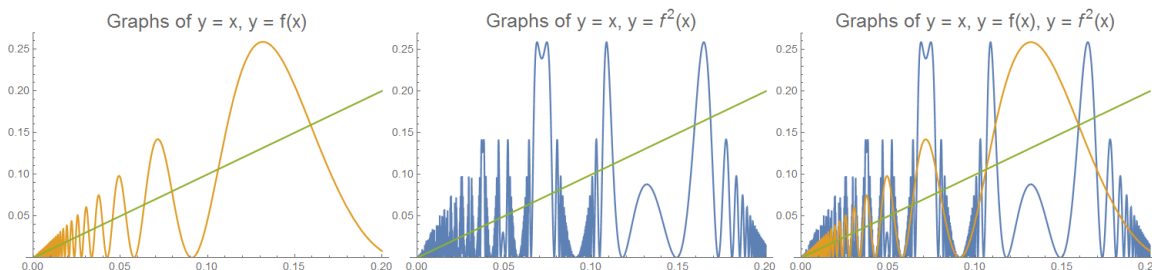
and observe that they both have a neutral fixed point at  $x_0 = 0$ .

- (a) Numerically compute the first five elements in the orbits of 0.1 and 0.01 under both  $s_1$  and  $s_2$  to at least 30 decimal places.
- (b) Based only on the orbits of 0.1 and 0.01, decide whether you think 0 is weakly attracting or weakly repelling for  $s_1$  and  $s_2$ .
- (c) Now compute the five elements in the orbit of 0.001 under  $s_1$  and  $s_2$  to 40 decimal places. Do the results agree with your guess from part (b)?
- (d) Show that the neutral fixed point is weakly repelling for one of  $s_1, s_2$  but weakly attracting for the other.

9. The goal of this problem is to explore a pathological example: a fixed point of a non-differentiable function.

Let  $f(x) = \begin{cases} x + x \sin\left(\frac{1}{x}\right) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ : note that  $f$  maps  $[0, \infty) \rightarrow [0, \infty)$ , and that  $f$  is continuous but not differentiable at  $x = 0$ .

- (a) Find all the fixed points of  $f$ , and show that (except for  $x = 0$ ) they are all repelling.
- (b) Numerically compute the first 10 elements in the orbits of each of  $x_0 = 0.1, 0.01, 0.001, 0.0001$ , and  $0.00015494157427205179$ . Are they being attracted to or repelled from zero in a consistent way?
- (c) Explain how to use the graphs of  $y = x, y = f(x)$ , and  $y = f^2(x)$  below to locate (i) fixed points, (ii) periodic points of order exactly 2, and (iii) points  $x_0$  such that  $f(x_0)$  is fixed but  $x_0$  is not fixed. (Do not do any computations.)



- (d) Show that, in any open interval  $(0, \epsilon)$  for any positive  $\epsilon$ , there are infinitely many points such that  $f(x) > x$ , infinitely many points such that  $f(x) = x$ , and infinitely many points such that  $f(x) < x$ . Explain why this makes it impossible to characterize the orbit behavior near 0 as “attracting” or “repelling”.