Contents

5 Topics in Hypothesis Testing

In this chapter, we study two important sampling distributions, the t distribution and the χ^2 distribution, and use them to extend the basic development of hypothesis testing from the previous chapter to make inferences about normally-distributed variables whose mean and standard deviation are unknown. Broadly speaking, our goal is to describe methods for making inferences about sampling data that is approximately normally distributed using only information derived from the sampling data itself.

We first discuss the t distributions: we motivate why they are necessary by explaining the additional difficulties that arise when the standard deviation must be estimated from the sample, and give methods for constructing confidence intervals for the mean when the standard deviation is unknown. We then describe how to perform t tests of various flavors, which allow us to test hypotheses about the mean of a normally-distributed quantity using sampling data.

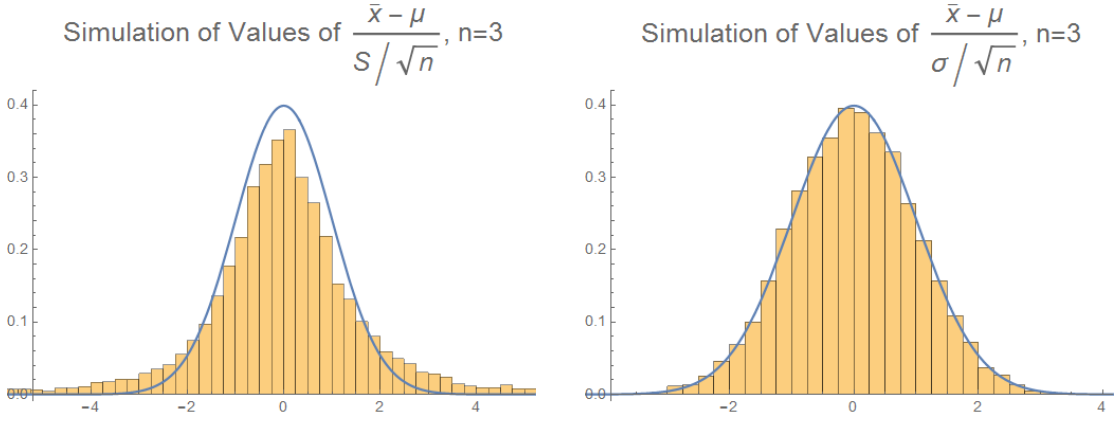
We then discuss the χ^2 distributions, and motivate how they arise when making inferences about the unknown variance of a normally-distributed random variable. We describe how to construct confidence intervals, and perform hypothesis tests, about the variance and standard deviation of a normally distributed variable. Finally, we discuss the χ^2 tests for goodness-of-fit and for independence, which allow us to assess the quality of statistical models and to assess the independence of random variables, respectively.

5.1 The t Distribution and t Tests

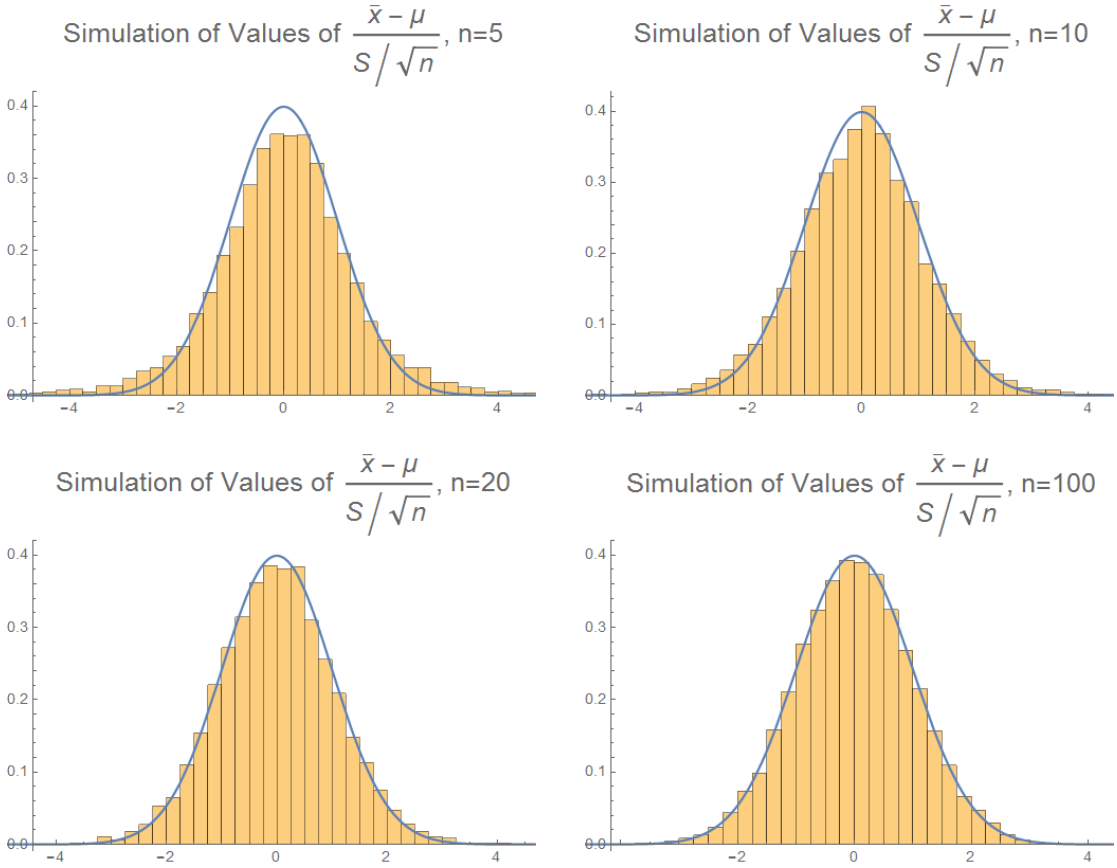
- Our goal in this section is to discuss the t distribution and t tests, which allow us to expand our hypothesis tests (and related discussion of confidence intervals) to approximately normally distributed quantities whose standard deviation is unknown.

5.1.1 The t Distribution

- In our discussion of hypothesis testing in the previous chapter, we relied on z tests, which require an approximately normally distributed test statistic whose standard deviation is known.
  - However, in most situations, it is unlikely that we would actually know the population standard deviation.
  - Instead, we must estimate the population standard deviation from the sample standard deviation.
  - As we have already discussed, for values $x_1, \ldots, x_n$ drawn from a normal distribution with unknown mean μ and unknown standard deviation σ, the maximum likelihood estimate $\hat{\sigma} = \sqrt{\frac{1}{n}\left[(x_1 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2\right]}$ for the standard deviation is biased. (Note that $\overline{x} = \frac{1}{n}(x_1 + \cdots + x_n)$ is the sample mean.)
  - Thus, instead of the estimator $\hat{\sigma}$, we use the sample standard deviation $S = \sqrt{\frac{1}{n-1}\left[(x_1 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2\right]}$, whose square S^2 is an unbiased estimator of the population variance σ^2.
- It may seem reasonable to say that if we use the estimated standard deviation S in place of the unknown population σ, then we should be able to use a z test with the resulting approximation, much as we did with the normal approximation to the binomial distribution.
  - However, this turns out not to be the case! We can make clearer why not by converting the discussion to a distribution with a single unknown parameter by working with the normalized ratio $\frac{\overline{x} - \mu}{S/\sqrt{n}}$, which has mean 0 and standard deviation 1 and is analogous to the z-score $\frac{\overline{x} - \mu}{\sigma/\sqrt{n}}$, whose distribution (under the assumptions of the null hypothesis) is the standard normal distribution of mean 0 and standard deviation 1.
  - If we take $\frac{\overline{x} - \mu}{S/\sqrt{n}}$ as our test statistic, then (as we will show) this test statistic is not normally distributed!
  - The distribution is similar in shape to the normal distribution, but it is in fact different, and is called the t distribution.
- We can illustrate visually the lack of normality of the normalized test statistic $\frac{\overline{x} - \mu}{S/\sqrt{n}}$ by simulating a sampling procedure.
  - Explicitly, suppose that X is normally distributed with mean $\mu = 0$ and standard deviation $\sigma = 1$, and we want to test the hypothesis that the mean actually is equal to 0 using the normalized test statistic $\frac{\overline{x} - \mu}{S/\sqrt{n}}$ with $n = 3$.
  - To understand the behavior of $\frac{\overline{x} - \mu}{S/\sqrt{n}}$, we sample the standard normal distribution to obtain 3 data points x_1, x_2, x_3 and then compute $\frac{\overline{x} - \mu}{S/\sqrt{n}}$ using the sample mean $\overline{x}$ and estimated standard deviation S.
  - The histogram below shows the results of performing this sampling 10000 times, along with the actual graph of the probability density function of the predicted normal distribution:



- Note how the actual histogram differs from the normal distribution: specifically, there are values occurring in the tails of the distribution far more often than they do for the normal distribution, while the values near the center occur slightly less often than predicted.
- In contrast, the same simulation using the test statistic $\dfrac{\overline{x}-\mu}{\sigma/\sqrt{n}}$ matches the normal distribution very closely, as can be seen in the second histogram above.
- The difference between the distribution of $\dfrac{\overline{x}-\mu}{S/\sqrt{n}}$ and the standard normal distribution is more pronounced when n is small. For larger n, the distribution looks much more approximately normal (this is related to the central limit theorem and the fact that the approximation of σ by S is unbiased in the limit as $n \to \infty$):



- What these plots indicate is that the actual distribution of the test statistic $\dfrac{\overline{x}-\mu}{S/\sqrt{n}}$ will depend on n, and that for larger n, it will be approximately normal.

- It is not a trivial matter to find the probability density function for the t distribution modeling the test statistic $\dfrac{\overline{x}-\mu}{S/\sqrt{n}}$.
  - The constants involved in the pdf involve a special function known as the gamma function, which generalizes the definition of the factorial function:

- Definition: If z is a positive real number[1], the gamma function $\Gamma(z)$ is defined to be the value of the improper integral $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx$.
  - The gamma function arises naturally in complex analysis, number theory, and combinatorics, in addition to our use here in statistics.

[1] In fact, this definition also makes perfectly good sense if z is a complex number whose real part is positive (which is why we used the letter z here).

- By integrating by parts, one may see that $\Gamma(z+1) = z\Gamma(z)$ for all z. Combined with the easy observation that $\Gamma(1) = 1$, we can see that $\Gamma(n) = (n-1)!$ for all positive integers n.
- The values of the gamma function at half-integers can also be computed explicitly: to compute $\Gamma(1/2)$ we may substitute $u = \sqrt{x}$ to see $\Gamma(1/2) = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}$ as we calculated before when analyzing the normal distribution.
- Then, by using the identity $\Gamma(z+1) = z\Gamma(z)$, we can calculate $\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\cdot\cdots\cdot\frac{1}{2}\sqrt{\pi} = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$.

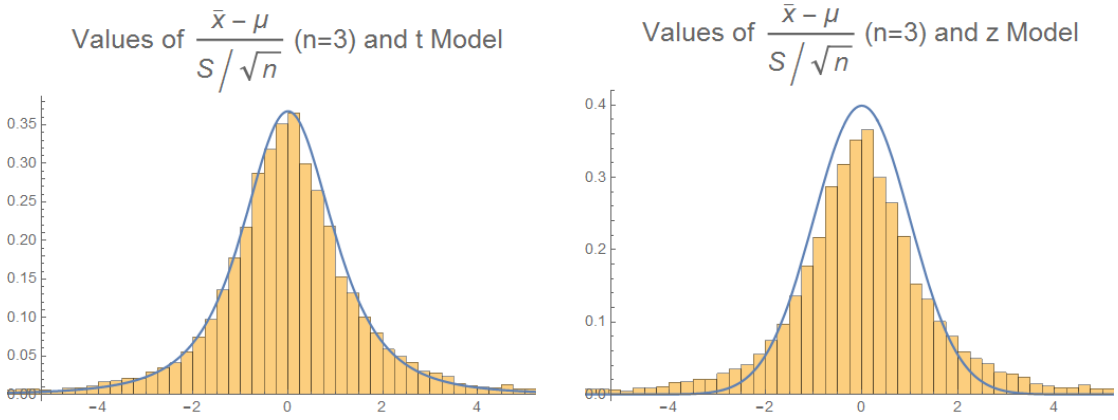
- Definition: The t distribution with k degrees of freedom is the continuous random variable T_k whose probability density function $p_{T_k}(x) = \frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})}\cdot(1+x^2/k)^{-(k+1)/2}$ for all real numbers x.
  - As we will outline in a moment, the t distribution with $n-1$ degrees of freedom is the proper model for the test statistic $\frac{\overline{x}-\mu}{S/\sqrt{n}}$.
  - This distribution was first derived in 1876 by Helmert and Lüroth, and then appeared in several other papers.
  - It is often referred to as Student's t distribution, because an analysis was published under the pseudonym "Student" by William Gosset, who because of his work at Guinness did not publish the results under his own name[2].
  - Example: The t distribution with 1 degree of freedom has probability density function $p_{T_1}(x) = \frac{1}{\pi(1+x^2)}$, which is the Cauchy distribution.
  - Example: The t distribution with 2 degrees of freedom has probability density function $p_{T_2}(x) = \frac{1}{(2+x^2)^{3/2}}$.
  - Example: The t distribution with 3 degrees of freedom has probability density function $p_{T_3}(x) = \frac{6\sqrt{3}}{\pi(3+x^2)^2}$.

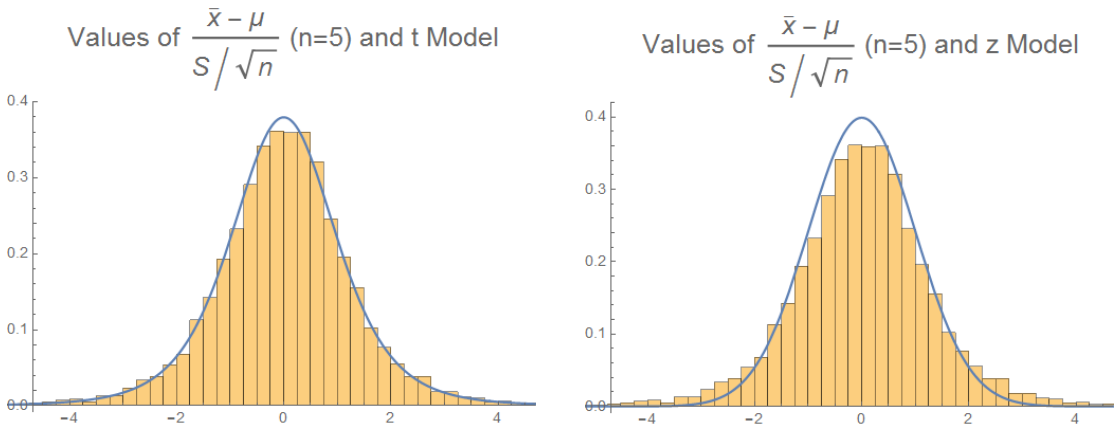
- We collect a few basic properties of the t distribution:
  - The probability density function of the t distribution is symmetric about 0, since $p_{T_k}(-x) = p_{T_k}(x)$.
  - Per the symmetry about 0, we would typically expect that the expected value of the distribution would be 0. This is true when $k \geq 2$, but in fact the expected value is undefined when $k = 1$ (as we noted previously, the expected value integral for the Cauchy distribution is a non-convergent improper integral).
  - It is more difficult to compute the variance, but by manipulating the integrals appropriately, one can eventually show that the variance is undefined for $k = 1$ (since the expected value is not defined), ∞ for $k = 2$, and $\frac{k}{k-2}$ for $k > 2$.
  - As $k \to \infty$, the probability density function $p_{T_k}(x)$ approaches the standard normal distribution: using the fact that $\lim_{k\to\infty}(1+y/k)^k = e^y$, we can see that $\lim_{k\to\infty}(1+x^2/k)^{-(k+1)/2} = e^{-x^2/2}$, and thus[3] $\lim_{k\to\infty}\frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})}\cdot(1+x^2/k)^{-(k+1)/2} = \lim_{k\to\infty}\frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})}\cdot\lim_{k\to\infty}(1+x^2/k)^{-(k+1)/2} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

[2]The standard version of the story holds that Guinness wanted all its staff to publish using pseudonyms to protect its brewing methods and related data, since a paper had been previously published by one of its statisticians that inadvertently revealed some of its trade secrets.

[3]We can compute the limit $\lim_{k\to\infty}\frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})}$ of the constant either using Stirling's approximation $k! \approx k^k e^{-k}\sqrt{2\pi k}$, which also extends to the gamma function, or simply by observing that it must be the constant that makes the resulting function a probability density function.

- Theorem (t Distribution): Suppose $n \geq 2$ and that $X_1, X_2, \ldots, X_n$ are independent, identically normally distributed random variables with mean μ and standard deviation σ. If $\overline{X} = \frac{1}{n}(X_1 + \cdots + X_n)$ denotes the sample mean and $S = \sqrt{\frac{1}{n-1}\left[(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2\right]}$ denotes the sample standard deviation, then the distribution of the normalized test statistic $\frac{\overline{X} - \mu}{S/\sqrt{n}}$ is the t distribution T_{n-1} with $n-1$ degrees of freedom.
    - We will only outline the full proof, since most of the actual calculations are lengthy and unenlightening.
    - Proof (outline): First, we show that the sample mean $\overline{X}$ and the sample standard deviation S are independent. This is relatively intuitive, but the proof requires the observation that orthogonal changes of variable preserve independence.
    - Next, we compute the probability density functions for the numerator $\overline{X} - \mu$ (which is normal with mean 0 and standard deviation $\sigma/\sqrt{n}$) and the denominator.
    - For the latter, we compute the probability density of $\frac{n-1}{\sigma^2}S^2 = \frac{1}{\sigma^2}\left[(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2\right] = (\frac{X_1-\mu}{\sigma})^2 + (\frac{X_2-\mu}{\sigma})^2 + \cdots + (\frac{X_n-\mu}{\sigma})^2$: this last expression can be shown to be equal to the sum of the squares of $n-1$ independent standard normal distributions, which is known as a χ^2 distribution (and which we will discuss in more detail later).
    - The pdf of the denominator $\frac{1}{S/\sqrt{n}}$ can then be computed using the pdf above, using standard techniques for computing the pdf of a function of a random variable.
    - Then, because since $\overline{X} - \mu$ and $S/\sqrt{n}$ were shown to be independent, the joint pdf for $\overline{X} - \mu$ and $S/\sqrt{n}$ is simply the product of their individual pdfs.
    - Then, at last, we can the probability density function for $\frac{\overline{X} - \mu}{S/\sqrt{n}} = (\overline{X} - \mu) \cdot \frac{1}{S/\sqrt{n}}$ can be calculated by evaluating an appropriate integral of the joint pdf of $\overline{X} - \mu$ and $S/\sqrt{n}$.
- We can illustrate this result using the sampled data from earlier.
    - Here are the same histograms with $n = 3$ and $n = 5$ as before, comparing the t distribution model to the normal distribution model:

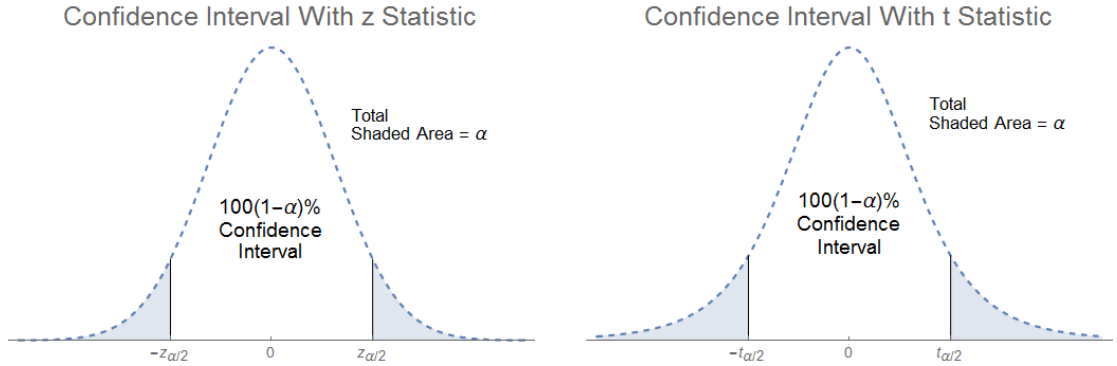




- It is quite obvious from the plots that the t distribution is a far superior model for these data samples.

5.1.2 Confidence Intervals Using t Statistics

- Before we discuss how to use the t distribution for hypothesis testing, we will mention how to use t statistics for finding confidence intervals.
  - The idea is quite simple: if we want to find a confidence interval for the unknown mean of a normal distribution whose standard deviation is also unknown, we can estimate the mean using the t distribution.
  - Specifically, since the normalized statistic $\frac{\overline{x}-\mu}{S/\sqrt{n}}$ is modeled by the t-distribution T_n with $n-1$ degrees of freedom, we can compute a $100(1-\alpha)\%$ confidence interval using a t-statistic in place of the z-statistic that we used for normally distributed random variables whose standard deviation was known:



  - Like with the normal distribution, we usually want to select the narrowest possible confidence interval, which will also be the one that is symmetric about our sample mean.
  - If we compute the constant $t_{\alpha/2,n}$ such that $P(-t_{\alpha/2,n} < T_{n-1} < t_{\alpha/2,n}) = 1-\alpha$, then this yields the $100(1-\alpha)\%$ confidence interval $\hat{\mu} \pm t_{\alpha/2,n} \cdot \frac{S}{\sqrt{n}} = (\hat{\mu} - t_{\alpha/2,n}\frac{S}{\sqrt{n}}, \hat{\mu} + t_{\alpha/2,n}\frac{S}{\sqrt{n}})$.
  - Using the symmetry of the t distribution, $P(-t_{\alpha/2,n} < T_{n-1} < t_{\alpha/2,n}) = 1-\alpha$ is equivalent to $P(T_{n-1} < -t_{\alpha/2,n}) = \alpha/2$, or also to $P(t_{\alpha/2,n} < T_{n-1}) = 1-(\alpha/2)$, which allows us to compute the value of $t_{\alpha/2,n}$ by evaluating the inverse cumulative distribution function for T_{n-1}.
  - We can summarize this discussion as follows:

- Proposition (t Confidence Intervals): A $100(1-\alpha)\%$ confidence interval for the unknown mean μ of a normal distribution with unknown standard deviation is given by $\hat{\mu} \pm t_{\alpha/2,n}\frac{S}{\sqrt{n}} = (\hat{\mu} - t_{\alpha/2,n}\frac{S}{\sqrt{n}}, \hat{\mu} + t_{\alpha/2,n}\frac{S}{\sqrt{n}})$ where n sample points $x_1, \ldots, x_n$ are taken from the distribution, $\hat{\mu} = \frac{1}{n}(x_1 + \cdots + x_n)$ is the sample mean, $S = \sqrt{\frac{1}{n-1}[(x_1-\hat{\mu})^2 + \cdots + (x_n-\hat{\mu})^2]}$ is the sample standard deviation, and $t_{\alpha/2,n}$ is the constant satisfying $P(-t_{\alpha/2,n} < T_{n-1} < t_{\alpha/2,n}) = 1-\alpha$.

- Some specific values of $t_{\alpha/2,n}$ for various common values of n and α are given in the table below (note that the last row for $n = \infty$ represents the entry for the normal distribution):

Entries give $t_{\alpha/2,n-1}$ such that $P(-t_{\alpha/2,n} < T_{n-1} < t_{\alpha/2,n}) = 1 - \alpha$								
n \ $1-\alpha$	50%	80%	90%	95%	98%	99%	99.5%	99.9%
2 (1 df)	1	3.0777	6.3138	12.706	31.820	63.657	127.32	636.62
3 (2 df)	0.8165	1.8856	2.9200	4.3027	6.9646	9.9248	14.089	31.599
4 (3 df)	0.7649	1.6377	2.3534	3.1824	4.5407	5.8409	7.4533	12.924
5 (4 df)	0.7407	1.5332	2.1318	2.7764	3.7469	4.6041	5.5976	8.6103
10 (9 df)	0.7027	1.3830	1.8331	2.2622	2.8214	3.2498	3.6897	4.7809
15 (14 df)	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768	3.3257	4.1405
20 (19 df)	0.6876	1.3277	1.7291	2.0930	2.5395	2.8609	3.1737	3.8834
50 (49 df)	0.6795	1.2991	1.6766	2.0096	2.4049	2.6800	2.9397	3.5004
100 (99 df)	0.6770	1.2902	1.6604	1.9842	2.3646	2.6264	2.8713	3.3915
∞	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758	2.8070	3.2905

- Example: The exam scores in a statistics class are expected to be normally distributed. 15 students' scores are sampled, and the average score is 78.2 points with a sample standard deviation of 9.1 points. Find 90%, 95%, and 99.5% confidence intervals for the true average score on the exam.
  - We have $\hat{\mu} = 78.2$ and $S = 9.1$, so the desired confidence interval is given by $\hat{\mu} \pm t_{\alpha/2,n}(S/\sqrt{n})$, where $n = 15$ here.
  - From the proposition and the table of values below it, we obtain the 90% confidence interval $\hat{\mu} \pm 1.7613 \cdot S/\sqrt{n} = \boxed{(74.06, 82.34)}$, the 95% confidence interval $\hat{\mu} \pm 2.1448 \cdot S/\sqrt{n} = \boxed{(73.16, 83.24)}$, and the 99% confidence interval $\hat{\mu} \pm 3.3257 \cdot S/\sqrt{n} = \boxed{(70.39, 86.01)}$.
- Example: A normal distribution with unknown mean and standard deviation is sampled five times, yielding the values 1.21, 4.60, 4.99, −2.21, and 3.21. Find 80%, 90%, 95%, and 99.9% confidence intervals for the true mean of the distribution. Compare the results to the corresponding confidence intervals for a normal distribution whose standard deviation is the same as this sample estimate.
  - First, we compute the sample mean $\hat{\mu} = \frac{1}{5}(1.21+4.60+4.99-2.21+3.21) = 2.36$ and the sample standard deviation $S = \sqrt{\frac{1}{4}[(1.21-2.36)^2 + (4.60-2.36)^2 + (4.99-2.36)^2 + (-2.21-2.36)^2 + (3.21-2.36)^2} = 2.9523$.
  - The desired confidence interval is given by $\hat{\mu} \pm t_{\alpha/2,n}(S/\sqrt{n})$, where $n = 5$ here.
  - From the proposition and the table of values below it, we obtain the 80% confidence interval $\hat{\mu} \pm 1.5332 \cdot S/\sqrt{n} = \boxed{(0.3357, 4.3843)}$, the 90% confidence interval $\hat{\mu} \pm 2.1318 \cdot S/\sqrt{n} = \boxed{(-0.4546, 5.1746)}$, the 95% confidence interval $\hat{\mu} \pm 2.7764 \cdot S/\sqrt{n} = \boxed{(-1.3057, 6.0257)}$, and the 99.9% confidence interval $\hat{\mu} \pm 8.6103 \cdot S/\sqrt{n} = \boxed{(-9.0083, 13.7283)}$.
  - The confidence interval estimates for a normal distribution are given by using the z-statistic (from the row with $n = \infty$) in place of the t-statistic.
  - We obtain the 80% confidence interval $\hat{\mu} \pm 1.2816 \cdot \sigma/\sqrt{n} = \boxed{(0.6679, 4.0521)}$, the 90% confidence interval $\hat{\mu} \pm 1.6449 \cdot \sigma/\sqrt{n} = \boxed{(0.1882, 4.5118)}$, the 95% confidence interval $\hat{\mu} \pm 1.9600 \cdot \sigma/\sqrt{n} = \boxed{(-0.2278, 4.9478)}$, and the 99.9% confidence interval $\hat{\mu} \pm 3.2905 \cdot \sigma/\sqrt{n} = \boxed{(-1.9845, 6.7045)}$.
  - Note how much narrower the normal confidence intervals are than the correct t confidence intervals, especially for the larger confidence percentages.
  - For example, if we erroneously quoted the 80% normal confidence interval, by using the cdf for the t distribution we can see that it is actually only a 64% confidence interval for the t statistic: quite a bit lower!
  - Similarly, if we erroneously quoted the 99.9% normal confidence interval, it would actually only be a 97% confidence interval for the t statistic.

- Example: To estimate the reaction yield, a new chemical synthesis is run three times, giving yields of 41.3%, 52.6%, and 56.1%. Find 50%, 80%, 90%, and 95% confidence intervals for the true reaction yield, under the assumption that the reaction yield is approximately normally distributed.
  - Since the reaction yield is approximately normally distributed, but we do not know the standard deviation, it is appropriate to use the t distribution here.
  - First, we compute the sample average $\hat{\mu} = \frac{1}{3}(41.3\% + 52.6\% + 56.1\%) = 50\%$, and the sample standard deviation $S = \sqrt{\frac{1}{2}\left[(41.3\% - 50\%)^2 + (52.6\% - 50\%)^2 + (56.1\% - 50\%)^2\right]} = 7.7350\%$.
  - Then the desired confidence interval is given by $\hat{\mu} \pm t_{\alpha/2,n}(S/\sqrt{n})$, where here $n = 3$.
  - From the proposition and the table of values below it, we obtain the 50% confidence interval $\hat{\mu} \pm 0.8165 \cdot S/\sqrt{n} = \boxed{(46.35\%, 53.65\%)}$, the 80% confidence interval $\hat{\mu} \pm 1.8856 \cdot S/\sqrt{n} = \boxed{(41.58\%, 58.42\%)}$, the 90% confidence interval $\hat{\mu} \pm 2.9200 \cdot S/\sqrt{n} = \boxed{(36.96\%, 63.04\%)}$, and the 95% confidence interval $\hat{\mu} \pm 4.3027 \cdot S/\sqrt{n} = \boxed{(30.79\%, 69.21\%)}$.

5.1.3 One-Sample t Tests

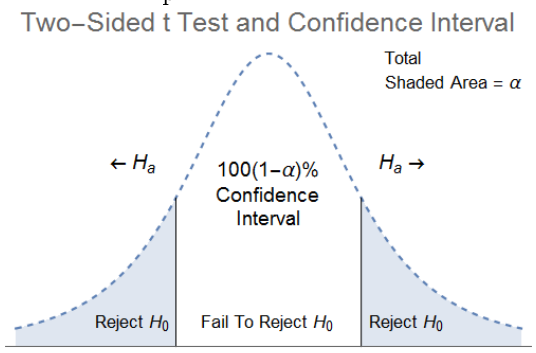
- In a similar way to how we adapted the procedure for constructing confidence intervals with the normal distribution to construct confidence intervals using t statistics, we can also adapt our procedures for z tests to do hypothesis testing with the t distribution: we call these t tests.
- We first describe one-sample t tests, in which we want to perform a hypothesis test on the unknown mean of a normal distribution with unknown standard deviation, based on an independent sampling of the distribution yielding n values $x_1, x_2, \ldots, x_n$.
  - The key difference here is that the standard deviation of the normal distribution is unknown, rather than given to us as is always the case with z tests.
  - As usual with hypothesis tests, we first select appropriate null and alternative hypotheses and a significance level α.
  - Our null hypothesis will be of the form H_0: $\mu = c$ for some constant c that is our hypothesized value for the mean of the normal distribution.
  - We take the test statistic $t = \dfrac{\overline{x} - \mu}{S/\sqrt{n}}$, where $\overline{x}$ is the sample mean and S is the sample standard deviation.
  - From our results about the t distribution, the distribution of the test statistic will be the t distribution T_{n-1} with $n - 1$ degrees of freedom.
  - We can then calculate the p-value based on the alternative hypothesis.
  - If the hypotheses are $H_0 : \mu = c$ and $H_a : \mu > c$, then the p-value is $P(T_{n-1} \geq t)$.
  - If the hypotheses are $H_0 : \mu = c$ and $H_a : \mu < c$, then the p-value is $P(T_{n-1} \leq t)$.
  - If the hypotheses are $H_0 : \mu = c$ and $H_a : \mu \neq c$, then the p-value is $P(|T_{n-1}| \geq |t|) = \begin{cases} 2P(T_{n-1} \geq t) & \text{if } t \geq \mu \\ 2P(T_{n-1} \leq t) & \text{if } t < \mu \end{cases}$.
  - We then compare the p-value to the significance level and then either reject or fail to reject the null hypothesis, as usual.
- Example: Suppose four values 9, 18, 7, 10 are sampled from a normal distribution with unknown mean and standard deviation. Test at the 20%, 11%, 2%, and 0.6% significance levels that the mean is (i) greater than 10, (ii) greater than 0, (iii) less than 25, (iv) less than 5, (v) equal to 10, and (vi) equal to 16.
  - First, we compute the sample mean $\hat{\mu} = \frac{1}{4}(9 + 18 + 7 + 10) = 11$ and sample standard deviation $S = \sqrt{\frac{1}{3}\left[(9 - 11)^2 + (18 - 11)^2 + (7 - 11)^2 + (10 - 11)^2\right]} = 4.8305$.
  - For (i), our hypotheses are $H_0 : \mu = 10$, $H_a : \mu > 10$; we want this one-sided alternative hypothesis since the actual sample mean is greater than 10.

- The value of our test statistic is $t = \dfrac{11-10}{4.8305/\sqrt{4}} = 0.4140$, giving p-value $P(T_{n-1} \geq 0.4140) = 0.3533$.
- Since this is greater than all four significance levels, we fail to reject the null hypothesis in all cases.
- For (ii), our hypotheses are $H_0 : \mu = 0$, $H_a : \mu > 0$; we want this one-sided alternative hypothesis since the actual sample mean is greater than 0.
- The value of our test statistic is $t = \dfrac{11-0}{4.8305/\sqrt{4}} = 4.5544$, giving p-value $P(T_{n-1} \geq 4.5544) = 0.00992$.
- Since this is less than the first three significance levels, we reject the null hypothesis in those cases. However, it is greater than 0.6%, so we fail to reject the null hypothesis at that significance level.
- For (iii), our hypotheses are $H_0 : \mu = 25$, $H_a : \mu < 25$; we want this one-sided alternative hypothesis since the actual sample mean is less than 25.
- The value of our test statistic is $t = \dfrac{11-25}{4.8305/\sqrt{4}} = -5.7966$, giving p-value $P(T_{n-1} \leq -5.7966) = 0.00511$.
- Since this is less than all four significance levels, we reject the null hypothesis in all cases.
- For (iv), our hypotheses are $H_0 : \mu = 5$, $H_a : \mu > 5$; we want this one-sided alternative hypothesis since the actual sample mean is greater than 5.
- The value of our test statistic is $t = \dfrac{11-5}{4.8305/\sqrt{4}} = 2.4842$, giving p-value $P(T_{n-1} \geq 2.4842) = 0.0445$.
- Since this is less than 20% and 11%, we reject the null hypothesis in those cases. However, it is greater than 2% and 0.6%, so we fail to reject the null hypothesis at those significance levels.
- For (v), our hypotheses are $H_0 : \mu = 10$, $H_a : \mu \neq 10$; we want this two-sided alternative hypothesis since we are only testing whether the mean equals 10 or not.
- The value of our test statistic is $t = \dfrac{11-10}{4.8305/\sqrt{4}} = 0.4140$, giving p-value $P(|T_{n-1}| \geq 0.4140) = 2P(T_{n-1} \geq 0.4140) = 0.7067$.
- Since this is (much!) greater than all of the listed significance levels, we fail to reject the null hypothesis in each case.
- For (vi), our hypotheses are $H_0 : \mu = 16$, $H_a : \mu \neq 16$; we want this two-sided alternative hypothesis since we are only testing whether the mean equals 16 or not.
- The value of our test statistic is $t = \dfrac{11-16}{4.8305/\sqrt{4}} = -2.0702$, giving p-value $P(|T_{n-1}| \geq |-2.0702|) = 2P(T_{n-1} \geq 2.0702) = 0.1302$.
- Since this is less than 20%, we reject the null hypothesis in that case. However, it is greater than 11%, 2% and 0.6%, so we fail to reject the null hypothesis at those significance levels.

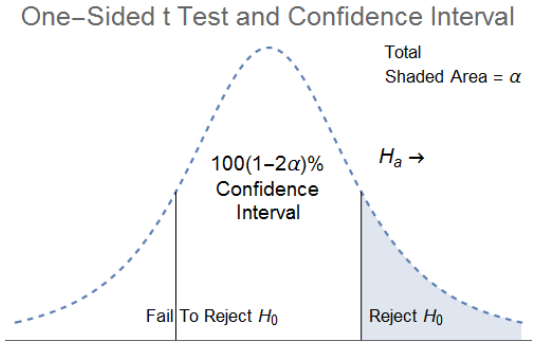
- Example: To estimate the reaction yield, a new chemical synthesis is run three times, giving yields of 41.3%, 52.6%, and 56.1%. It is expected that the yield should be approximately normally distributed. Test at the 20%, 8%, and 1% significance levels the hypotheses that (i) the average yield is above 45%, (ii) the average yield is below 57%, (iii) the average yield is above 64%.
  - First, we compute the sample average $\hat{\mu} = \frac{1}{3}(41.3\% + 52.6\% + 56.1\%) = 50\%$, and the sample standard deviation $S = \sqrt{\dfrac{1}{2}\left[(41.3\% - 50\%)^2 + (52.6\% - 50\%)^2 + (56.1\% - 50\%)^2\right]} = 7.7350\%$.
  - For (i), our hypotheses are $H_0 : \mu = 45\%$, $H_a : \mu > 45\%$; we want this one-sided alternative hypothesis since the actual sample mean is greater than 45%.
  - The value of our test statistic is $t = \dfrac{50\% - 45\%}{7.7350\%/\sqrt{3}} = 1.1196$, giving p-value $P(T_{n-1} \geq 1.1196) = 0.1896$.
  - Since this is less than the first significance level 20%, we reject the null hypothesis in that case. However, it is greater than 8% and 1%, so we fail to reject the null hypothesis at those significance levels.
  - For (ii), our hypotheses are $H_0 : \mu = 57\%$, $H_a : \mu < 57\%$; we want this one-sided alternative hypothesis since the actual sample mean is less than 57%.

- The value of our test statistic is $t = \dfrac{50\% - 57\%}{7.7350\%/\sqrt{3}} = -1.5675$, giving p-value $P(T_{n-1} \leq -1.5675) = 0.1288$.
- Since this is less than the first significance level 20%, we reject the null hypothesis in that case. However, it is greater than 8% and 1%, so we fail to reject the null hypothesis at those significance levels.
- For (iii), our hypotheses are $H_0 : \mu = 64\%$, $H_a : \mu < 64\%$; we want this one-sided alternative hypothesis since the actual sample mean is less than 60%.
- The value of our test statistic is $t = \dfrac{50\% - 64\%}{7.7350\%/\sqrt{3}} = -3.1349$, giving p-value $P(T_{n-1} \leq -3.1349) = 0.0442$.
- Since this is less than the first two significance levels 20% and 8%, we reject the null hypothesis in those cases. However, it is greater than 1%, so we fail to reject the null hypothesis at that significance level.

- Just as with z tests, we can also interpret one-sample t tests using confidence intervals. The idea is exactly the same as before, except the underlying distribution is now a t distribution rather than a normal distribution.
  - Since we work with the normalized test statistic, we have to compare to the corresponding normalized confidence interval, which is $(-t_{\alpha/2,n}, t_{\alpha/2,n})$.
  - For a two-sided alternative hypothesis, if we give a $100(1-\alpha)\%$ confidence interval around the mean of a distribution under the conditions of the null hypothesis, then we will reject the null hypothesis with significance level α precisely when the sample statistic lies outside the normalized confidence interval:



  - We can do the same thing with a one-sided alternative hypothesis, but because of the lack of symmetry in the rejection region, we instead need to use a $100(1-2\alpha)\%$ confidence interval to get the correct area:



- Example: The online list prices for four randomly-chosen statistics textbooks are \$193.95, \$171.89, \$221.80, and \$215.32. Assuming that the prices of statistics textbooks are approximately normally distributed, find 80%, 90%, 95%, 98%, 99%, and 99.5% confidence intervals for the average list price of a statistics textbook online. Then test at the 10% and 1% significance levels the hypotheses that (i) the average price is \$200, (ii) the average price is \$230, (iii) the average price is \$275, (iv) the average price is above \$170, (v) the average price is above \$270.

- The sample mean is $\hat{\mu} = \frac{1}{4}(\$193.95 + \$171.89 + \$221.80 + \$215.32) = \$200.74$ with sample standard deviation $S = \sqrt{\frac{1}{3}\left[(\$193.95 - \$200.74)^2 + (\$171.89 - \$200.74)^2 + (\$221.80 - \$200.74)^2 + (\$215.32 - \$200.74)^2\right]} = \$22.617$.
- For the confidence levels, we look up or calculate the appropriate t-statistics for the given confidence levels and $n = 4$ (3 degrees of freedom).
- The confidence intervals are as follows:

α	80%	90%	95%	98%	99%	99.5%
Conf Int ($)	(182.22, 219.26)	(174.13, 227.35)	(164.75, 236.73)	(149.39, 252.09)	(134.69, 266.79)	(116.46, 285.02)

- For the hypothesis tests, we just need to identify whether or not the hypothesized average is in the appropriate confidence interval (depending on the alternative hypothesis).
- For (i), we take $H_0 : \mu = 200$, $H_a : \mu \neq 200$. This is a two-sided confidence interval, and so we want to look at the $100(1-\alpha)\%$ confidence intervals for $\alpha = 0.10$ and $\alpha = 0.01$. Since 200 lies in both the 90% and 99% confidence intervals, we fail to reject the null hypothesis in both cases.
- Explicitly, the value of the normalized test statistic is $t = \dfrac{\$200.74 - \$200}{\$22.617/\sqrt{3}} = 0.0654$, and so our p-value is $2P(T_{n-1} \geq 0.0654) = 0.9519$: well above the 10% significance level.
- For (ii), we take $H_0 : \mu = 230$, $H_a : \mu \neq 230$. As before, we want to look at the 90% and 99% confidence intervals.
- Since 230 lies outside the 90% confidence interval, we reject the null hypothesis at the 10% significance level. But since 230 lies inside the 99% confidence interval, we fail to reject the null hypothesis at the 1% significance level.
- Explicitly, the value of the normalized test statistic is $t = \dfrac{\$200.74 - \$230}{\$22.617/\sqrt{3}} = -2.5875$, and so our p-value is $2P(T_{n-1} \leq -2.5875) = 0.0813$: below the 10% significance level but above the 1% significance level.
- For (iii), we take $H_0 : \mu = 275$, $H_a : \mu \neq 275$. As before, we want to look at the 90% and 99% confidence intervals. Since 275 lies outside both the 90% and 99% confidence intervals, we reject the null hypothesis in both cases.
- Explicitly, the value of the normalized test statistic is $t = \dfrac{\$200.74 - \$275}{\$22.617/\sqrt{3}} = -6.5669$, and so our p-value is $2P(T_{n-1} \leq -6.5669) = 0.00718$: below the 1% significance level.
- For (iv), we take $H_0 : \mu = 170$, $H_a : \mu > 170$. Now we have a one-sided alternative hypothesis, so we want to look at the $100(1-2\alpha)\%$ confidence interval. Since 170 lies below the 80% confidence interval, we reject the null hypothesis at the 10% significance level. However, 170 does lie inside the 98% confidence interval, so we fail to reject the null hypothesis at the 1% significance level.
- Explicitly, the value of the normalized test statistic is $t = \dfrac{\$200.74 - \$170}{\$22.617/\sqrt{3}} = 2.7184$, and so our p-value is $P(T_{n-1} \geq 2.7184) = 0.02653$: below the 10% significance level but above the 1% significance level.
- For (v), based on the statement we could try taking $H_0 : \mu = 270$, $H_a : \mu > 270$. As above, we want to look at the $100(1-2\alpha)\%$ confidence interval.
- Notice that 270 lies outside both the 80% and 98% confidence intervals. However, the confidence intervals themselves are below 270, meaning that our deviation away from the hypothesized value falls into the null hypothesis tail of the distribution, rather than the alternative hypothesis tail.
- Thus, we fail to reject the null hypothesis at either the 10% significance level or the 1% significance level.
- Explicitly, the value of the normalized test statistic is $t = \dfrac{\$200.74 - \$270}{\$22.617/\sqrt{3}} = -6.1247$, and so our p-value is $P(T_{n-1} \geq -6.1247) = 0.9956$.
- Here, we actually ought to have tested the alternative hypothesis $H_a : \mu < 270$, since the sample mean was less than 270. In this case, 270 would still lie outside both the 80% and 98% confidence intervals, but now 270 would land in the alternative hypothesis tail rather than the null hypothesis tail, so we would reject the null hypothesis at both significance levels.
- In that situation, the p-value is $P(T_{n-1} \leq -6.1247) = 0.00438$, which is indeed quite small.

5.1.4 Two-Sample t Tests

- Now that we have treated the situation of one-sample t tests, we discuss the thornier issue of two-sample t tests, in which we want to compare the unknown means of two normally-distributed populations with unknown standard deviations.
- Let us first review the setup for a two-sample z test:
  - Suppose the two populations are labeled A and B, with respective means μ_A and μ_B and population standard deviations σ_A and σ_B. We sample population A a total of n_A times, and population B a total of n_B times.
  - Like with two-sample z tests, we would like to take our test statistic as the difference $\mu_A - \mu_B$ in the two population means.
  - First suppose that we are testing whether $\mu_A = \mu_B$, which we can equivalently phrase as asking whether $\mu_A - \mu_B = 0$.
  - Then, per our assumptions, the sample mean $\hat{\mu}_A$ will be normally distributed with mean μ_A and standard deviation $\sigma_A/\sqrt{n_A}$, while $\hat{\mu}_B$ will be normally distributed with mean μ_B and standard deviation $\sigma_B/\sqrt{n_B}$.
  - The key piece of information here is that since $\hat{\mu}_A$ and $\hat{\mu}_B$ are independent and normally distributed, their difference is also normally distributed with mean $\mu_A - \mu_B$ and standard deviation $\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}$.
  - Therefore, if we are testing the null hypothesis H_0: $\mu_A - \mu_B = c$, then under the assumption of the null hypothesis, our test statistic $\hat{\mu}_A - \hat{\mu}_B$ will be normally distributed with mean c and standard deviation $\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}$.
- Now we analyze the situation of a two-sample t test, in which we do not know the standard deviations σ_A and σ_B.
  - If we do not know σ_A and σ_B, then we must use the sample standard deviation estimates S_A and S_B to estimate the standard deviation of the quantity $\hat{\mu}_A - \hat{\mu}_B$.
  - However, just as we discussed before, using the sample standard deviation in place of the population standard deviation changes the underlying distributions: although $\frac{\hat{\mu}_A - \mu_A}{\sigma_A/\sqrt{n_A}}$ has the standard normal distribution, $\frac{\hat{\mu}_A - \mu_A}{S_A/\sqrt{n_A}}$ has the distribution of the random variable T_{n_A-1}.
  - If we solve for the distribution of $\hat{\mu}_A$, we see it is no longer given by the normal random variable $\mu_A + \frac{\sigma}{\sqrt{n_A}}N_{0,1}$ (normal with with mean μ_A and standard deviation $\sigma_A/\sqrt{n_A}$), but rather a "rescaled" t distribution $\mu_A + \frac{S_A}{\sqrt{n_A}}T_{n_A-1}$.
  - Likewise, the random variable $\hat{\mu}_B$ has a rescaled t distribution $\mu_B + \frac{S_B}{\sqrt{n_B}}T_{n_B-1}$.
  - Then the quantity $\hat{\mu}_A - \hat{\mu}_B$ is modeled by the random variable $\left[\mu_A + \frac{S_A}{\sqrt{n_A}}T_{n_A-1}\right] - \left[\mu_B + \frac{S_B}{\sqrt{n_B}}T_{n_B-1}\right] = (\mu_A - \mu_B) + \frac{S_A}{\sqrt{n_A}}T_{n_A-1} - \frac{S_B}{\sqrt{n_B}}T_{n_B-1}$.
  - Equivalently, $\mu_A - \mu_B$ is modeled by the random variable $(\hat{\mu}_A - \hat{\mu}_B) + \frac{S_A}{\sqrt{n_A}}T_{n_A-1} - \frac{S_B}{\sqrt{n_B}}T_{n_B-1}$, where we absorbed the minus sign into the two t-distributions (since they are symmetric about 0).
- The problem here is that we do not have a nice description of what the difference between two (scaled) t distributions looks like.
  - For normal distributions, we can use the very convenient fact that the sum or difference of normal random variables is also normal; that is not the case for t distributions!

- In principle, because we know the probability density functions of T_{n_A-1} and T_{n_B-1}, and they are independent, we could calculate the probability density function of the random variable listed above for particular values of all of the parameters.
- But that does not solve our problem, because we need to write down a test statistic whose distribution is independent of the test parameters (i.e., that does not depend on S_A and S_B, which is not the case in the expression above).
- In general, solving this problem of finding an appropriate statistic for testing the equality of two sample means from normally distributed random samples is known as the Behrens-Fisher problem. (Various generalizations are also often named with this moniker as well.)

- The first approximation is to construct a pooled standard deviation, much like our approach previously when we did two-sample z tests for binomially distributed data. The approach comes from the following theorem:

- Theorem (t Distribution With Pooled Variance): Suppose that $X_1, \ldots, X_n$ are independent and identically normally distributed with mean μ_X and standard deviation σ, and that $Y_1, \ldots, Y_m$ are independent and identically normally distributed with mean μ_Y and standard deviation σ. If $\hat{\mu}_X$, $\hat{\mu}_Y$, S_X, and S_Y denote the sample means and sample standard deviations of $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$, then for the pooled standard deviation $S_{\text{pool}} = \sqrt{\dfrac{(n-1)S_X^2 + (m-1)S_Y^2}{m+n-2}} = \sqrt{\dfrac{\sum_{i=1}^n (X_i - \hat{\mu}_X)^2 + \sum_{j=1}^m (Y_j - \hat{\mu}_Y)^2}{m+n-2}}$,: the distribution of the test statistic $\dfrac{(\hat{\mu}_X - \hat{\mu}_Y) - (\mu_X - \mu_Y)}{S_{\text{pool}}\sqrt{\dfrac{1}{n} + \dfrac{1}{m}}}$ is T_{m+n-2}, the t distribution with $m+n-2$ degrees of freedom.
  - The idea of the proof is similar to the theorem we proved earlier for the probability density function of the t distribution, and we will in fact reduce to applying the arguments of that theorem in a special case.
  - Proof (outline): Observe that the test statistic is the quotient of $\dfrac{(\hat{\mu}_X - \hat{\mu}_Y) - (\mu_X - \mu_Y)}{\sigma\sqrt{\dfrac{1}{n} + \dfrac{1}{m}}}$ by $\dfrac{S_{\text{pool}}}{\sigma}$.
  - The first term $\dfrac{(\hat{\mu}_X - \hat{\mu}_Y) - (\mu_X - \mu_Y)}{\sigma\sqrt{\dfrac{1}{n} + \dfrac{1}{m}}}$, from our discussion above, is normally distributed with mean 0 and standard deviation 1 (i.e., its distribution is simply the standard normal).
  - The other term has square $\dfrac{S_{\text{pool}}^2}{\sigma^2} = \dfrac{1}{m+n-2}\left[\sum_{i=1}^n \left(\frac{X_i - \hat{\mu}_X}{\sigma}\right)^2 + \sum_{j=1}^m \left(\frac{Y_j - \hat{\mu}_Y}{\sigma}\right)^2\right]$.
  - As noted in the proof of our earlier theorem, the sum $\sum_{i=1}^n \left(\frac{X_i - \hat{\mu}_X}{\sigma}\right)^2$ can be rewritten as the sum of squares of $n-1$ standard normal variables, and by the same argument, $\sum_{i=1}^n \left(\frac{X_i - \hat{\mu}_X}{\sigma}\right)^2$ can be rewritten as the sum of squares of $m-1$ standard normal variables.
  - Therefore, the sum of these terms is the sum of squares of $m+n-2$ standard normal variables.
  - But as we showed in our earlier theorem, the distribution of a ratio $\dfrac{\overline{x} - \mu}{S/\sqrt{n}} = \dfrac{(\overline{x} - \mu)/(\sigma/\sqrt{n})}{S/\sigma} = \dfrac{N_{0,1}}{S/\sigma}$ is the t distribution T_{n-1} with degrees of freedom equal to the number of squares of standard normal variables summed in the denominator.
  - Since there are $m+n-2$ standard normal variables, that means that our quotient of $\dfrac{(\hat{\mu}_X - \hat{\mu}_Y) - (\mu_X - \mu_Y)}{\sigma\sqrt{\dfrac{1}{n} + \dfrac{1}{m}}}$ by $\dfrac{S_{\text{pool}}}{\sigma}$ is t-distributed with $m+n-2$ degrees of freedom, as claimed.

- This theorem gives us an explicit procedure for performing a two-sample t test with a pooled standard deviation where the population variances are assumed to be equal. This test is known as Student's equal-variances t test.
  - First, we select appropriate null and alternative hypotheses and a significance level α.
  - Our null hypothesis will be of the form H_0: $\mu_A - \mu_B = c$ for some constant c that is our hypothesized value for the difference of the means (usually 0).

- We take the test statistic $t = \dfrac{(\hat{\mu}_A - \hat{\mu}_B) - c}{S_{\text{pool}}\sqrt{\dfrac{1}{n_A} + \dfrac{1}{n_B}}}$, where $S_{\text{pool}} = \sqrt{\dfrac{(n_A-1)S_A^2 + (n_B-1)S_B^2}{n_A + n_B - 2}}$ is the pooled standard deviation estimate.
- From our theorem above, the distribution of the test statistic will be the t-distribution $T_{n_A+n_B-2}$ with $n_A + n_B - 2$ degrees of freedom.
- We can then calculate the p-value based on the alternative hypothesis.
- If the hypotheses are $H_0 : \mu_A - \mu_B = c$ and $H_a : \mu_A - \mu_B > c$, then the p-value is $P(T_{n_A+n_B-2} \geq t)$.
- If the hypotheses are $H_0 : \mu_A - \mu_B = c$ and $H_a : \mu_A - \mu_B < c$, then the p-value is $P(T_{n_A+n_B-2} \leq t)$.
- If the hypotheses are $H_0 : \mu_A - \mu_B = c$ and $H_a : \mu_A - \mu_B \neq c$, then the p-value is $P(|T_{n_A+n_B-2}| \geq |t|) = \begin{cases} 2P(T_{n_A+n_B-2} \geq t) & \text{if } t \geq \mu \\ 2P(T_{n_A+n_B-2} \leq t) & \text{if } t < \mu \end{cases}$.
- We then compare the p-value to the significance level and then either reject or fail to reject the null hypothesis, as usual.

- Before doing an example, we can also give some brief motivation for why these particular choices of parameters (the pooled standard deviation, the test statistic, and the number of degrees of freedom) are logical.
  - The denominator of the test statistic is analogous to the standard deviation $\sqrt{\dfrac{\sigma^2}{n_A} + \dfrac{\sigma^2}{n_B}}$ for the difference of the two normal distributions, and is what we would divide by if we were doing a two-sample z test to get a normalized statistic.
  - The pooled standard deviation can be thought of as arising from the pooled variance: if the two sample means were actually equal to the same number μ, then the pooled variance of the set $\{X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_m\}$ would be $\dfrac{1}{m+n}\left[(X_1-\mu)^2 + \cdots + (X_n-\mu)^2 + (Y_1-\mu)^2 + \cdots + (Y_m-\mu)^2\right]$.
  - However, since the two sets don't have the same mean, we instead measure them relative to their own means. Furthermore, the resulting variance estimate is biased (for the same reason that the estimate for the sample variance for a single sample is biased), so we must divide by $\dfrac{1}{m+n-2}$ rather than $\dfrac{1}{m+n}$ to unbias it.
  - We can then rewrite the complicated sum above more simply in terms of the sample standard deviations S_X and S_Y: this is precisely the pooled standard deviation S_{pool}.
  - The number of degrees of freedom of the t distribution is $m+n-2$, because we have $m+n$ data points, but we lose one degree of freedom by comparing A to its mean, and we lose another by comparing B to its mean.

- Example: A statistics instructor wants to determine whether students do better on exams in a morning class or in an evening class. They randomly sample 11 exams from the morning class, which have an average score of 84 and a sample standard deviation of 13, and compare to a random sample of 11 exams from the evening class, which have an average score of 77 and a sample standard deviation of 9. Assuming that the population variances are equal, test at the 11%, 3%, and 0.7% significance levels the hypotheses that (i) the average score in the morning class is higher, (ii) the average score in the two classes are different, and (iii) the average score in the morning class is at least 2 points higher than the evening class.
  - Since the population variances are assumed to be equal, we use Student's equal-variances t test.
  - For (i), our hypotheses are $H_0 : \mu_m - \mu_e = 0$, $H_a : \mu_m - \mu_e > 0$; we want this one-sided alternative hypothesis since the morning class average is higher than the evening class average.
  - The pooled standard deviation is $S_{\text{pool}} = \sqrt{\dfrac{(11-1)\cdot 13^2 + (11-1)\cdot 9^2}{11+11-2}} = 11.1803$.
  - The test statistic is $t = \dfrac{(84-77)-0}{11.1803 \cdot \sqrt{\dfrac{1}{11} + \dfrac{1}{11}}} = 1.4683$, giving p-value $P(T_{20} \geq 1.4683) = 0.07878$.

- Since the p-value is less than the first significance level, we reject the null hypothesis in that case. However, it is greater than the other two significance levels, so we fail to reject the null hypothesis in those cases.
- For (ii), our hypotheses are $H_0 : \mu_m - \mu_e = 0$, $H_a : \mu_m - \mu_e \neq 0$; we want this two-sided alternative hypothesis since now we want only to test whether the scores are equal.
- The parameters are the same as in (i) above; the only difference is that the p-value is now $2P(T_{20} \geq 1.4683) = 0.1576$.
- Since the p-value is above all three significance levels, we fail to reject the null hypothesis in each case.
- For (iii), our hypotheses are $H_0 : \mu_m - \mu_e = 2$, $H_a : \mu_m - \mu_e > 2$; we want this one-sided alternative hypothesis since the morning class actually did score more than two points above the evening class.
- The pooled standard deviation is the same as before.
- The test statistic is $t = \dfrac{(84-77)-2}{11.1803 \cdot \sqrt{\dfrac{1}{11}+\dfrac{1}{11}}} = 1.0488$, giving p-value $P(T_{20} \geq 1.0488) = 0.1534$.
- Since the p-value is above all three significance levels, we fail to reject the null hypothesis in each case.

- In most situations when we are comparing two populations, it is not reasonable to assume that the population variances are the same. For this reason, various unpooled two-sample t tests have been developed.
  - The most popular such test is known as Welch's unequal-variances t test. It is generally more accurate than Student's equal-variances t test (described above) in the situation where the two sample variances are far apart, or when the sample sizes differ drastically.
  - With null hypothesis H_0: $\mu_A - \mu_B = c$, the test statistic is $t = \dfrac{(\hat{\mu}_A - \hat{\mu}_B) - c}{S_{\text{unpool}}}$, where $S_{\text{unpool}} = \sqrt{\dfrac{S_A^2}{n_A} + \dfrac{S_B^2}{n_B}}$ is the natural standard deviation estimate for the difference in the sample means.
  - As we discussed earlier, the resulting test statistic does not actually have an exact distribution we can describe in any convenient way.
  - However, as proven by Welch, it is approximately t-distributed by the t distribution with the number of degrees of freedom equal to the rather complicated formula $df = \dfrac{(S_A^2/n_A + S_B^2/n_B)^2}{\frac{1}{n_A-1}(S_A^2/n_A)^2 + \frac{1}{n_B-1}(S_B^2/n_B)^2}$.
  - Most computer systems allow the number of degrees of freedom to be an arbitrary positive real number (in which case one may use the exact value given above); otherwise, such as when using tables, one usually rounds to the nearest integer.

- Welch's result is quite technical, but we can describe roughly where the formula for the degrees of freedom comes from.
  - The idea is to rewrite the quotient in the test statistic (in the same way we did in the theorem earlier) and try to write the denominator ratio as the sum of squares of independent standard normals.
  - This cannot be done exactly, but if it could, we would then be able to find the number of terms by using the method of moments to compare the means and variances of the two expressions.
  - Carefully going through the calculations eventually yields the degree-of-freedom formula given above.

- We can use Welch's unequal-variances t test to compare the sample means for populations whose variances are not assumed to be equal.

- Example: Use Welch's unequal-variances t test with the previous example (morning class with 11 exams of average score 84 and sample standard deviation 13, evening class with 11 of average score 77 and sample standard deviation 9) to test at the 11%, 3%, and 0.7% significance levels the hypotheses that (i) the average score in the morning class is higher, (ii) the average score in the two classes are different, and (iii) the average score in the morning class is at least 2 points higher than the evening class.

- For (i), our hypotheses are $H_0 : \mu_m - \mu_e = 0$, $H_a : \mu_m - \mu_e > 0$; we want this one-sided alternative hypothesis since the morning class average is higher than the evening class average.
- The unpooled standard deviation is $S_{\text{unpool}} = \sqrt{\frac{13^2}{11} + \frac{9^2}{11}} = 4.7673$.
- The test statistic is $t = \frac{(84-77)-0}{4.7673} = 1.4683$, and the number of degrees of freedom is $df = \frac{(13^2/11 + 9^2/11)^2}{\frac{1}{11-1}(13^2/11)^2 + \frac{1}{11-1}(9^2/11)^2} = 17.7951$ giving p-value $P(T_{17.7951} \geq 1.4683) = 0.07973$.
- Since the p-value is less than the first significance level, we reject the null hypothesis in that case. However, it is greater than the other two significance levels, so we fail to reject the null hypothesis in those cases.
- Remark: Note from before that the pooled p-value estimate was 0.07878, which is quite close.
- For (ii), our hypotheses are $H_0 : \mu_m - \mu_e = 0$, $H_a : \mu_m - \mu_e \neq 0$; we want this two-sided alternative hypothesis since now we want only to test whether the scores are equal.
- The parameters are the same as in (i) above; the only difference is that the p-value is now $2P(T_{17.7951} \geq 1.4683) = 0.1594$.
- Since the p-value is above all three significance levels, we fail to reject the null hypothesis in each case.
- For (iii), our hypotheses are $H_0 : \mu_m - \mu_e = 2$, $H_a : \mu_m - \mu_e > 2$; we want this one-sided alternative hypothesis since the morning class actually did score more than two points above the evening class.
- The unpooled standard deviation and degrees of freedom are the same as before.
- The test statistic is $t = \frac{(84-77)-2}{4.7673} = 1.0488$, giving p-value $P(T_{17.7951} \geq 1.0488) = 0.1542$.
- Since the p-value is above all three significance levels, we fail to reject the null hypothesis in each case.
- Remark: Note from before that the pooled p-value estimate was 0.1534, which is again quite close.

- We can also adapt the two testing methods to give confidence intervals for the difference of two sample means.
  - Using either Student's or Welch's procedure, we simply compute the appropriate t-statistic for the t distribution with the number of degrees of freedom indicated by the method, and take as the standard deviation either $S = S_{\text{pool}}\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$ or $S = S_{\text{unpool}}$ respectively.
  - Then the desired $100(1-\alpha)\%$ confidence interval will be given by $(\hat{\mu}_A - \hat{\mu}_B) \pm t_{\alpha/2,df} S$.

- Example: The salaries of six randomly-chosen male faculty in a university's math department are \$51000, \$90500, \$46000, \$97000, \$108000, \$85000 and the salaries of five randomly-chosen female faculty in the same department are \$56600, \$55000, \$104000, \$70500, \$87000. Test at the 20%, 11%, and 2% significance levels whether the average salary of male faculty is equal to the average salary of female faculty using (i) both Student's equal-variances t test and (ii) Welch's unequal-variances t test. Also, find 80% and 95% confidence intervals for the difference between the average salaries of male and female faculty.
  - For the male faculty ($n_m = 6$) the sample mean is $\hat{\mu}_m = \$79583$ and the sample standard deviation is $S_m = \$25315$, while for the female faculty ($n_f = 5$) the sample mean is $\hat{\mu}_f = \$74620$ and the sample standard deviation is $S_f = \$20875$.
  - Our hypotheses are $H_0 : \mu_m - \mu_f = 0$ and $H_a : \mu_m - \mu_f \neq 0$.
  - For (i), the pooled standard deviation is $S_{\text{pool}} = \sqrt{\frac{(6-1)\cdot \$25315^2 + (5-1)\cdot \$20875^2}{6+5-2}} = \$23446$.
  - The test statistic is $t = \frac{(\$79583 - \$74620) - 0}{\$23446 \cdot \sqrt{\frac{1}{6} + \frac{1}{5}}} = 0.3496$, giving p-value $2P(T_9 \geq 0.3496) = 0.7347$.
  - The p-value is quite large so we fail to reject the null hypothesis in all cases.
  - For (ii), the unpooled standard deviation is $S_{\text{unpool}} = \sqrt{\frac{\$25315^2}{6} + \frac{\$20875^2}{5}} = \$13927$.

- The test statistic is $t = \dfrac{(\$79583 - \$74620) - 0}{\$13927} = 0.3564$, and the number of degrees of freedom is $df = \dfrac{(\$25315^2/6 + \$20875^2/5)^2}{\frac{1}{6-1}(\$25315^2/6)^2 + \frac{1}{5-1}(\$20875^2/5)^2} = 8.9991$ giving p-value $P(T_{8.9991} \geq 0.3564) = 0.7298$.
- The p-value is again quite large so we fail to reject the null hypothesis in all cases.
- To compute the 80% and 95% confidence intervals, we need to compute the appropriate t-statistics.
- We see that the difference in the average salaries is $\hat{\mu}_m - \hat{\mu}_f = \$4963$.
- For the pooled estimate, there are 9 degrees of freedom, so using a t table or computer yields $t_{\alpha/2,n} = 1.3830$ for the 80% confidence interval and $t_{\alpha/2,n} = 2.2622$.
- Then $S = S_{\text{pool}}\sqrt{\dfrac{1}{6} + \dfrac{1}{5}} = \$14197$, so our 80% confidence interval is $\$4963 \pm 1.3830 \cdot \$14197 = (-\$14672, \$24598)$ and our 95% confidence interval is $\$4963 \pm 2.2622 \cdot \$14197 = (-\$27153, \$37079)$.
- For the unpooled estimate, there are $df = 8.9991$ degrees of freedom, so using a t table or computer yields $t_{\alpha/2,n} = 1.3830$ for the 80% confidence interval and $t_{\alpha/2,n} = 2.2622$. (The degrees of freedom are so close to 9 in this case it actually doesn't matter if we just round to 9.)
- Then $S = S_{\text{unpool}} = \$13927$, so our 80% confidence interval is $\$4963 \pm 1.3830 \cdot \$14197 = (-\$14298, \$24225)$ and our 95% confidence interval is $\$4963 \pm 2.2622 \cdot \$14197 = (-\$26542, \$36469)$.
- We can see here that the two estimates are quite close, since the sample variances are not far away from each other.

- We make a few brief remarks about when to use these various t tests.
  - Most sources still identify Student's t test as the preferred test to use when the sample variances are not far away from each other, and give various approximate rules for deciding what "far away" means (e.g., requiring the variances not to differ by a factor of more than 2).
  - When the sample variances are far apart, Welch's t test tends to give more reliable results in the sense of having lower type I and type II error probabilities. Even when the sample variances are close, Welch's t test is generally not that much worse than Student's t test, which has a higher power in the situations where it should be used.
  - Neither test is exact (in the sense that it gives exact p-values) except in the case of Student's t test where the population variances are equal. In practice, this means that the type I error rate will deviate somewhat from the desired significance level α.
  - Welch's t test tends to maintain a type I error rate closer to the desired significance level α than Student's t test does, although of course there are scenarios in which it is worse.
  - It is also worth noting that, as the sample sizes of both groups become large, both tests are very close to the two-sample z test we have previously described. In practice, with samples larger than 100-200 or so, there is a negligible difference between the results of these t tests and the simpler z test.

- We also mention one additional scenario involving t tests and the comparison of two samples, involving matched pairs.
  - In matched-pairs comparisons, we are comparing the means of two sets of paired data.
  - A common situation is to make a before-and-after comparison of measurements taken before applying a treatment to measurements taken after applying the treatment: the goal is to determine whether (and how) the treatment affected the average outcome.
  - Although this scenario involves two data sets, the matched-pairs design means that the initial and later measurements will be correlated, so it is not appropriate to use a two-sample t test.
  - Instead, what we do is compute the difference in the results (for each individual), and use a one-sample t test to compare the average outcome to 0.

- Example: To test whether studying improves students' exam scores, an instructor has 6 students take a pre-assessment, complete a study module, and then a post-assessment. The results are summarized in the table

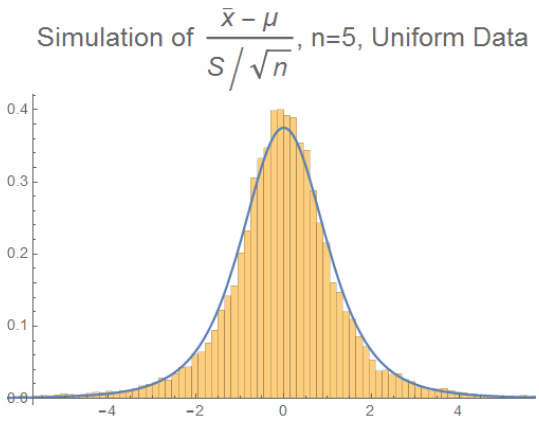
below. Test, at the 9%, 1%, and 0.3% significance levels whether the students' scores improved after studying:

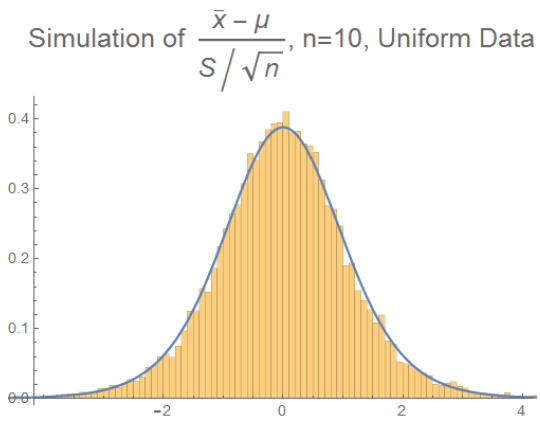
Student	A	B	C	D	E	F
Pre-study	61	71	90	81	55	81
Post-study	74	88	97	80	85	93

- Here, we have matched-pair data, because the measurements of the scores are coming from the same students. Since the values in the samples are not independent, but come from matched pairs, we want to use a one-sample t test here.
- Our hypotheses are $H_0 : \mu_{\text{post}} = \mu_{\text{pre}}$ and $H_a : \mu_{\text{post}} > \mu_{\text{pre}}$, which we can rephrase in terms of the difference in means $\mu_{\text{diff}} = \mu_{\text{post}} - \mu_{\text{pre}}$ as H_0: $\mu_{\text{diff}} = 0$ and $H_a : \mu_{\text{diff}} > 0$.
- Our test statistic is the difference in means μ_{diff}, which will be t-distributed with $6 - 1 = 5$ degrees of freedom.
- Our sample data set consists of the six differences of scores $\{13, 17, 7, -1, 30, 12\}$, with mean $\hat{\mu}_{\text{diff}} = 13$ and sample standard deviation $S = 10.3730$, and the value of the sample statistic is $t = \dfrac{\hat{\mu}_{\text{diff}} - 0}{S/\sqrt{n}} = 3.0698$.
- Thus, the p-value is $P(T_5 \geq 3.0698) = 0.01389$.
- Since the p-value is below 9% we reject the null hypothesis at that significance level, but since it is above 1% and 0.3% we fail to reject at those significance levels.

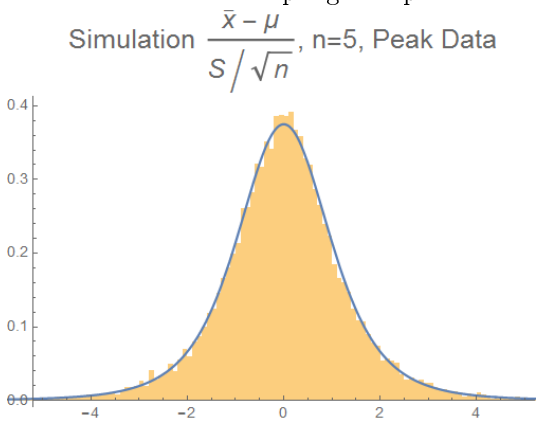
5.1.5 Robustness of t Tests

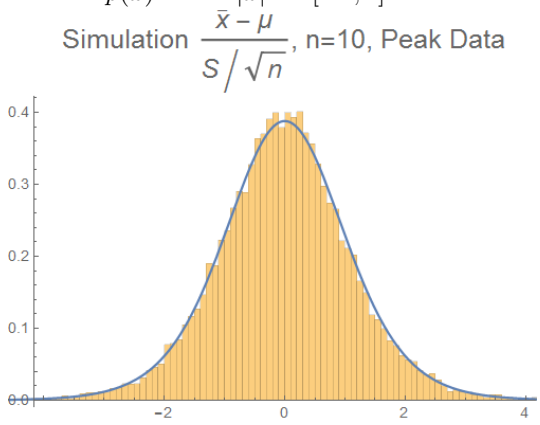
- We will make a few comments about robustness: the accuracy of the tests when applied to distributions that are not exactly the ones predicted by the model.
  - All of our discussion of z tests and t tests has been predicated on the assumption that the underlying populations we are studying are normally distributed.
  - In reality, except for very rare examples arising in physics with phenomena having exact theoretical models, no population is precisely normally distributed.
  - It is therefore important to understand how well the tests we have developed will perform in situations where the underlying distributions are not exactly normal, but only approximately normal.
  - It is a similar concern to the one that motivated our discussion of the t distribution and t tests: we could simply have tried using z tests but with S in place of σ. The resulting test would then not be exact, but we could hope that it is fairly close.
  - As we have explained, with small samples using a z place instead of a t test will generally be much less accurate (in the sense that the type I and type II error probabilities will generally be much larger).
  - However, with large samples (e.g., n around 100 or more) then the difference between the standard normal distribution and the t distribution is negligible, and so using a z test in place of a t test in such situations does not introduce much error.
  - In principle, if we had a different underlying distribution (e.g., a uniform distribution), we could develop analogues of the z test and t test, and in fact there are many other statistical tests that have been developed precisely to allow accurate study of data sets that have very non-normally-shaped distributions.
- It turns out that the t test is actually fairly robust, in that it performs fairly well even with distributions that are moderately non-normal.
  - Here are some examples for different simulations of the t-statistic $\dfrac{\overline{x} - \mu}{S/\sqrt{n}}$ for sampling the uniform distribution on $[-1, 1]$:



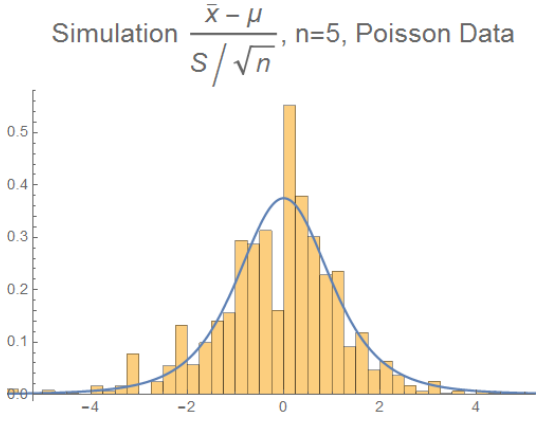


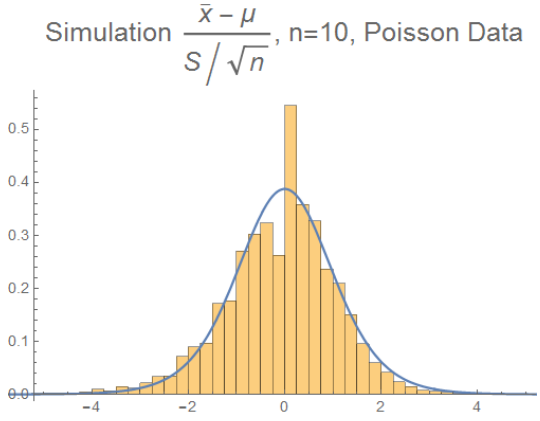
- Here are the results for sampling the "peak" distribution with $p(x) = 1 - |x|$ on $[-1, 1]$:



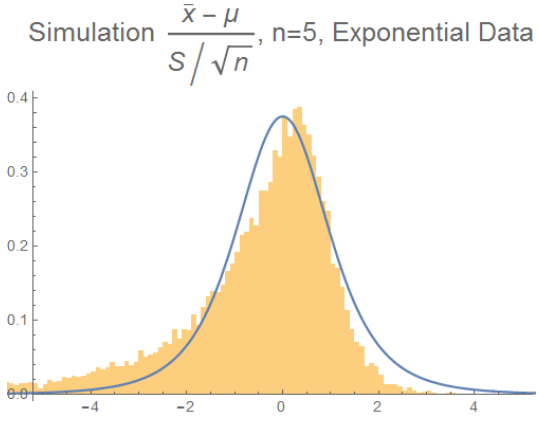


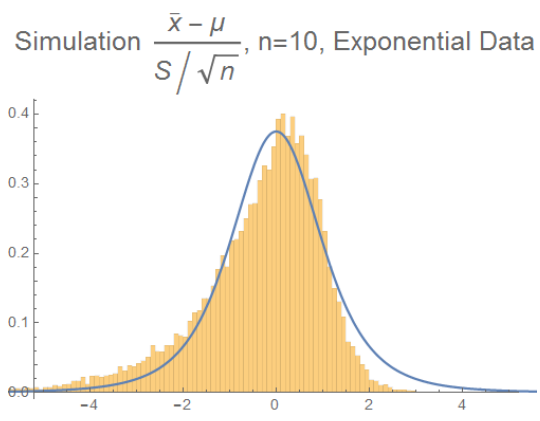
- Here are the results for sampling the Poisson distribution with parameter $\lambda = 3$:





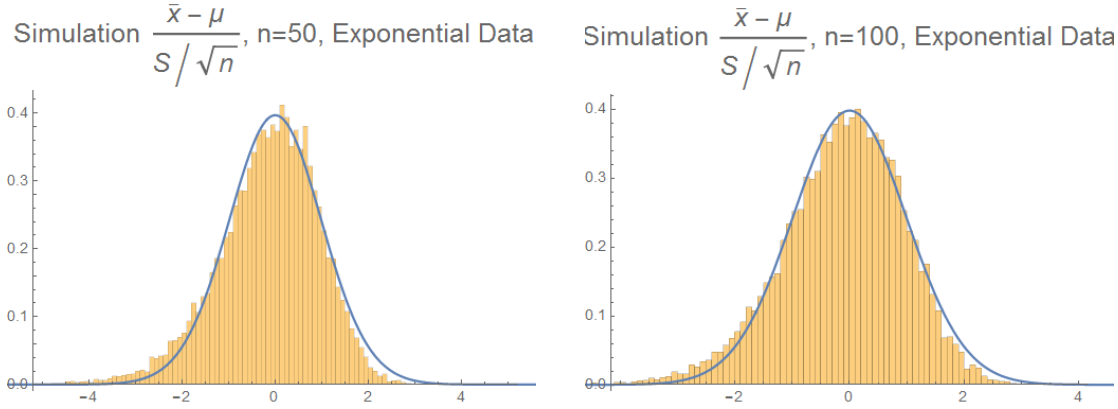
- Here are the results for sampling the exponential distribution with parameter $\lambda = 1/2$:





- We can see from the simulations that the t distribution is fairly close for the uniform and peak distributions, it is off a bit for the Poisson, and it is very far off for the exponential.
- The uniform and peak distributions are both symmetric and do not have wide tails.
- The Poisson distribution is more skewed and has a wider tail. It also has the difficulty that it is discrete and that small samples will sometimes yield all identical values (giving a sample standard deviation of 0, yielding an undefined test statistic): this explains the peculiar spike at 0.
- The exponential distribution is very skewed, which causes the resulting test statistic also to be skewed. We can see that the t distribution is not a very good model here even with a sample size $n = 10$.

- In general, the t distribution models the sample statistic $\dfrac{\overline{x} - \mu}{S/\sqrt{n}}$ well when the underlying distribution is symmetric, but not as well when the underlying distribution is asymmetric or skewed to one side.
  - When the underlying distribution is asymmetric or skewed, using a t test will not generally give reliable results with small sample sizes, and it is necessary to use different tests that are more robust for skewed data.
  - With large sample sizes (the exact definition of large, of course, depends on the scenario, but as we have seen in our discussion of the central limit theorem, usually $n = 100 - 200$ or so is quite sufficient), the central limit theorem will eventually take over and cause the sample average to be approximately normally distributed, even if the original distribution was asymmetric or skewed.
  - In such cases, since the t distribution is so close to the normal distribution, either the t test or the z test will be fairly reliable.
  - For example, here are the results of simulating the test statistic for larger n with exponentially distributed data:



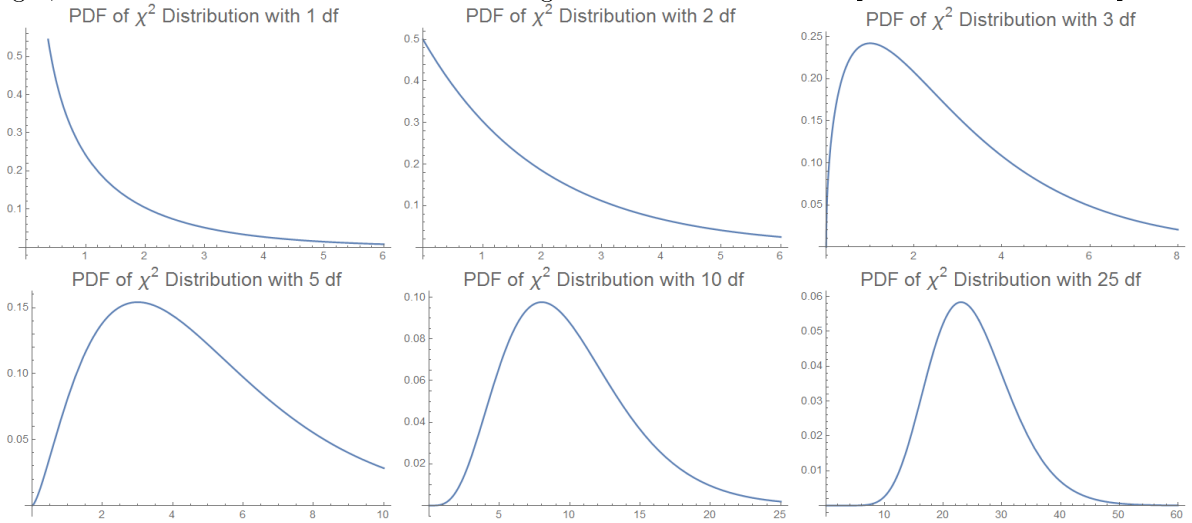
  - We can see here that although there is still some skewness in the histogram, it is now better approximated by the t distribution.

5.2 The χ^2 Distribution and χ^2 Tests

- Our goal in this section is to discuss the χ^2 distribution and three different χ^2 tests, which allow us to expand our hypothesis tests to testing statements about the variance (and standard deviation) of a distribution.
  - All of our hypothesis tests so far have essentially focused on testing statements about the mean of a distribution.
  - However, in certain scenarios, some of which we will discuss now, we might also want to test hypotheses about the variance of a distribution.
  - If the underlying distribution is normal, or obtained as a sum of normal distributions, we can use the χ^2 distribution to construct such tests.

5.2.1 The χ^2 Distribution

- We have previously discussed (at length) methods for constructing confidence intervals for the mean μ of a normally-distrbuted random variable with (known or unknown) standard deviation σ, given a random sample $x_1, \ldots, x_n$ from this normal distribution.
- Our present goal is to apply the same ideas to construct confidence intervals for the variance σ^2 (or equivalently the standard deviation σ) of the normal distribution.
  - Of course, the problem is only interesting when we do not already know σ, which is to say, when we are estimating it from the sample.
  - As we have also discussed at length, the sample variance $S^2 = \dfrac{1}{n-1}\left[(x_1 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2\right]$ gives an unbiased estimator for σ^2.
  - In order to construct confidence intervals for σ^2, it is enough to write down the underlying distribution of the statistic $\dfrac{(n-1)S^2}{\sigma^2} = \left(\dfrac{x_1 - \overline{x}}{\sigma}\right)^2 + \cdots + \left(\dfrac{x_n - \overline{x}}{\sigma}\right)^2$.
- Definition: The χ^2 distribution with k degrees of freedom is the continuous random variable Q_k whose probability density function $p_{Q_k}(x) = \dfrac{1}{2^{k/2}\Gamma(k/2)} \cdot x^{(k/2)-1}e^{-x/2}$ for all real numbers $x > 0$.
  - As we will outline in a moment, the χ^2 distribution with $n-1$ degrees of freedom is the proper model for the test statistic $\dfrac{(n-1)S^2}{\sigma^2}$.
  - Example: The χ^2 distribution with 1 degree of freedom has probability density function $p_{Q_1}(x) = \dfrac{1}{\sqrt{2\pi x}}e^{-x/2}$ for $x > 0$.
  - Example: The χ^2 distribution with 2 degrees of freedom has probability density function $p_{Q_1}(x) = \dfrac{1}{2}e^{-x/2}$ for $x > 0$, which is the exponential distribution with parameter $\lambda = 1/2$.
  - Example: The χ^2 distribution with 3 degrees of freedom has probability density function $p_{Q_1}(x) = \dfrac{\sqrt{x}}{\sqrt{2\pi}}e^{-x/2}$ for $x > 0$.
  - It is not hard to show using the probability density function that the χ^2 distribution with k degrees of freedom has mean k and variance $2k$.
  - We also emphasize that the χ^2 distribution, unlike the normal and t distributions, is quite skewed to the right, but the skewness decreases with more degrees of freedom. Here are plots of some of these pdfs:



- Proposition (χ^2 Distribution From Normals): If $X_1, \ldots, X_n$ are independent standard normal random variables (i.e., with mean 0 and standard deviation 1), then the random variable $Q_n = X_1^2 + \cdots + X_n^2$ has a χ^2 distribution with n degrees of freedom.
  - The proof is a relatively straightforward calculation using the joint pdf of $X_1, \ldots, X_n$ (which is simply the product of the one-variable pdfs, since these variables are independent).
  - We then just have to set up and evaluate the appropriate n-dimensional integral to compute the probability density function of $Q_n = X_1^2 + \cdots + X_n^2$.
  - We will omit the explicit details of the calculations, although we will mention that the main idea in the computation of the integral is to convert to n-dimensional spherical coordinates.
  - As a corollary, since the χ^2 distribution is obtained by summing independent, identically-distributed random variables, by the central limit theorem it approaches the appropriate normal distribution (with the same mean and variance) as $k \to \infty$.
- Theorem (χ^2 Distribution As Sampling Distribution): Suppose $n \geq 2$ and that $X_1, X_2, \ldots, X_n$ are independent, identically normally distributed random variables with mean μ and standard deviation σ. If $\overline{X} = \frac{1}{n}(X_1 + \cdots + X_n)$ denotes the sample mean and $S^2 = \frac{1}{n-1}\left[(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2\right]$ denotes the sample variance, then the distribution of the test statistic $\frac{(n-1)S^2}{\sigma^2}$ is the χ^2 distribution Q_{n-1} with $n-1$ degrees of freedom.
  - Proof: Let $W = \sum_{i=1}^n \left[\frac{X_i - \mu}{\sigma}\right]^2$. Then

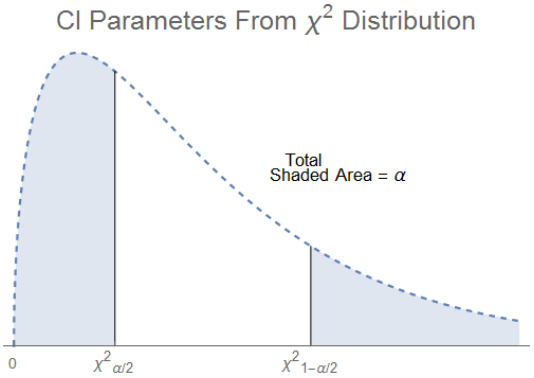
$$W = \sum_{i=1}^n \left[\frac{(X_i - \overline{X}) + (\overline{X} - \mu)}{\sigma}\right]^2 = \sum_{i=1}^n \left[\frac{X_i - \overline{X}}{\sigma}\right]^2 + 2\sum_{i=1}^n \left[\frac{X_i - \overline{X}}{\sigma}\right]\left[\frac{\overline{X} - \mu}{\sigma}\right] + \sum_{i=1}^n \left[\frac{\overline{X} - \mu}{\sigma}\right]^2.$$

  - In this last expression, the first term is $\frac{(n-1)S^2}{\sigma^2}$, the middle term is zero by evaluating the sum (since $\sum_{i=1}^n X_i = \sum_{i=1}^n \overline{X}$), and the last term is $n\left[\frac{\overline{X} - \mu}{\sigma}\right]^2 = \left[\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right]^2$. Thus, we see that $W = \frac{(n-1)S^2}{\sigma^2} + \left[\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right]^2$.
  - Note that W is the sum of squares of n independent standard normal variables, so it has a χ^2 distribution with n degrees of freedom.
  - Also, S and $\overline{X}$ are independent (as we previously noted in our derivation of the properties of the t distribution).
  - Thus, since $\left[\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right]^2$ is the square of a standard normal variable, and S is independent from it, this means the distribution of $\frac{(n-1)S^2}{\sigma^2} = W - \left[\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}\right]^2$ is given by the sum of squares of $n-1$ independent standard normal variables[4].
  - This means $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2 distribution with $n-1$ degrees of freedom, as claimed.

5.2.2 χ^2 Confidence Intervals and Hypothesis Tests

- The theorem above tells us that we can use the χ^2 distribution as a model for the ratio between the sample variance and the population variance, after rescaling appropriately.

[4] Technically, this step requires additional justification: one may make this argument completely precise using moment-generating functions.

- Thus, we can construct confidence intervals for the population variance using χ^2-statistics and the sample variance.
- Specifically, since the statistic $\dfrac{(n-1)S^2}{\sigma^2}$ is modeled by the χ^2 distribution Q_{n-1} with $n-1$ degrees of freedom, we can compute a $100(1-\alpha)\%$ confidence interval using χ^2-statistics in place of the z- and t-statistics that we used for the confidence intervals for the mean of a normally distributed random variable.
- 

- Here, we want the parameters $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ to satisfy $P(Q_{n-1} \le \chi^2_{\alpha/2}) = \alpha/2 = P(Q_{n-1} \ge \chi^2_{1-\alpha/2})$, so that the total area in each tail of the distribution is α, leaving an area $1-\alpha$ in the middle.
- In other words, we have $P(\chi^2_{\alpha/2} \le Q_{n-1} \le \chi^2_{1-\alpha/2}) = 1-\alpha$. Since $\dfrac{(n-1)S^2}{\sigma^2}$ is χ^2-distributed, this is equivalent to saying that $P(\chi^2_{\alpha/2} \le \dfrac{(n-1)S^2}{\sigma^2} \le \chi^2_{1-\alpha/2}) = 1-\alpha$.
- We can then rewrite the above equation to get the desired $100(1-\alpha)\%$ confidence interval for σ:

- Proposition (χ^2 Confidence Intervals): A $100(1-\alpha)\%$ confidence interval for the unknown variance σ^2 of a normal distribution with unknown mean and standard deviation is given by $\left(\dfrac{(n-1)S^2}{\chi^2_{1-\alpha/2,n}}, \dfrac{(n-1)S^2}{\chi^2_{\alpha/2,n}}\right)$ where n sample points $x_1, \ldots, x_n$ are taken from the distribution, $\hat{\mu} = \frac{1}{n}(x_1 + \cdots + x_n)$ is the sample mean, $S = \sqrt{\frac{1}{n-1}[(x_1-\hat{\mu})^2 + \cdots + (x_n-\hat{\mu})^2]}$ is the sample standard deviation, and $\chi^2_{\alpha/2,n}$ and $\chi^2_{1-\alpha/2,n}$ are the constants satisfying $P(Q_{n-1} \le \chi^2_{\alpha/2,n-1}) = \alpha/2 = P(Q_{n-1} \ge \chi^2_{1-\alpha/2,n-1})$ where Q_{n-1} is χ^2-distributed with $n-1$ degrees of freedom.
  - For a $100(1-\alpha)\%$ confidence interval for σ we just take the square root: $\left(\sqrt{\dfrac{n-1}{\chi^2_{1-\alpha/2,n-1}}}S, \sqrt{\dfrac{n-1}{\chi^2_{\alpha/2,n-1}}}S\right)$.
  - In order to compute the necessary χ^2 statistics, we must (as with the normal distribution or t distribution) either use a table of values or a computer to evaluate the inverse cumulative distribution function. Here is a small table of such values:

Inverse-CDF entries give $\chi^2_{\beta,n}$ such that $P(Q_n < \chi^2_{\beta,n}) = \beta$.										
df	0.005	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99	0.995
1	0.0000	0.0002	0.0010	0.0039	0.0158	2.7055	3.8415	5.0239	6.6349	7.8794
2	0.0100	0.0201	0.0506	0.1026	0.2107	4.6052	5.9915	7.3778	9.2103	10.5966
3	0.0717	0.1148	0.2158	0.3518	0.5844	6.2514	7.8147	9.3484	11.3449	12.8382
4	0.2070	0.2971	0.4844	0.7107	1.0636	7.7794	9.4877	11.1433	13.2767	14.8603
5	0.4117	0.5543	0.8312	1.1455	1.6103	9.2364	11.0705	12.8325	15.0863	16.7496
6	0.6757	0.8721	1.2373	1.6354	2.2041	10.6446	12.5916	14.4494	16.8119	18.5476
7	0.9893	1.2390	1.6899	2.1673	2.8331	12.0170	14.0671	16.0128	18.4753	20.2777
8	1.3444	1.6465	2.1797	2.7326	3.4895	13.3616	15.5073	17.5345	20.0902	21.9550
9	1.7349	2.0879	2.7004	3.3251	4.1682	14.6837	16.9190	19.0228	21.6660	23.5894
10	2.1559	2.5582	3.2470	3.9403	4.8652	15.9872	18.3070	20.4832	23.2093	25.1882
15	4.6009	5.2293	6.2621	7.2609	8.5468	22.3071	24.9958	27.4884	30.5779	32.8013
20	7.4338	8.2604	9.5908	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968

- We need to compute both $\chi^2_{\alpha/2,n}$ and $\chi^2_{1-\alpha/2,n}$, since the χ^2 distribution is not symmetric.

- Example: A normal distribution is sampled six times yielding values -3, 1, 5, -2, 7, and 8. Find 80%, 90%, and 99% confidence intervals for the standard deviation of the distribution.
  - We first compute the sample mean $\mu = 2.6667$ and sample standard deviation $S = 4.6762$.
  - Since there are 6 values, the number of degrees of freedom for the underlying χ^2 statistics is 5.
  - For the 80% confidence interval, the required values are $\chi^2_{0.9,5} = 9.2364$ and $\chi^2_{0.1,5} = 1.6103$, and so the confidence interval for σ is $\left(\sqrt{\frac{5}{9.2364}} \cdot 4.6762, \sqrt{\frac{5}{1.6103}} \cdot 4.6762\right) = \boxed{(3.4405, 8.2400)}$.
  - For the 90% confidence interval, the required values are $\chi^2_{0.95,5} = 11.0705$ and $\chi^2_{0.05,5} = 1.1455$, and so the confidence interval for σ is $\left(\sqrt{\frac{5}{11.0705}} \cdot 4.6762, \sqrt{\frac{5}{1.1455}} \cdot 4.6762\right) = \boxed{(3.1426, 9.7697)}$.
  - For the 99% confidence interval, the required values are $\chi^2_{0.995,5} = 16.7496$ and $\chi^2_{0.005,5} = 0.4117$, and so the confidence interval for σ is $\left(\sqrt{\frac{5}{16.7496}} \cdot 4.6762, \sqrt{\frac{5}{0.4117}} \cdot 4.6762\right) = \boxed{(2.5549, 16.2962)}$.

- We can also adapt our characterization to give a procedure for doing a hypothesis test about the unknown variance of a normal distribution based on an independent sampling of the distribution yielding n values $x_1, x_2, \ldots, x_n$.
  - As usual with hypothesis tests, we first select appropriate null and alternative hypotheses and a significance level α.
  - Our null hypothesis will be of the form H_0: $\sigma^2 = c$ for some constant c, with an appropriate one-sided or two-sided alternative hypothesis.
  - We take the test statistic $\chi^2 = \dfrac{(n-1)S^2}{c}$, where S is the sample standard deviation.
  - From our results about the χ^2 distribution, the test statistic is χ^2-distributed with $n-1$ degrees of freedom.
  - If the test is one-sided, we can calculate the p-value based on the alternative hypothesis.
  - If the hypotheses are $H_0 : \sigma^2 = c$ and $H_a : \sigma^2 > c$, then the p-value is $P(Q_{n-1} \geq \chi^2)$.
  - If the hypotheses are $H_0 : \sigma^2 = c$ and $H_a : \sigma^2 < c$, then the p-value is $P(Q_{n-1} \leq \chi^2)$.
  - If the hypotheses are $H_0 : \sigma^2 = c$ and $H_a : \sigma^2 \neq c$, then it is not as obvious how to compute a p-value because of the asymmetry of the χ^2 distribution. We will take the convention of doubling the appropriate one-sided tail probability (as we did with z tests and t tests).
  - We then compare the p-value to the significance level and then either reject or fail to reject the null hypothesis, as usual.

- Example: A normal distribution is sampled six times yielding values -3, 1, 5, -2, 7, and 8. Test at the 10% and 1% significance levels the hypothesis that the variance of this distribution is (i) greater than 16, and (ii) less than 225.
  - We calculated the sample standard deviation $S = 4.6762$ earlier, and the number of degrees of freedom is still 5.
  - For (i), our hypotheses are $H_0 : \sigma^2 = 16$ and $H_a : \sigma^2 > 16$, since in fact the sample standard deviation is greater than 16.
  - Our test statistic is $\chi^2 = \dfrac{(n-1)S^2}{\sigma^2} = \dfrac{5 \cdot 4.6762^2}{16} = 6.8333$, and so the p-value is $P(Q_5 > 6.8333) = 0.2333$.
  - Since the p-value is greater than both significance levels, we fail to reject the null hypothesis in both cases.

- This result is reasonable, since the sample variance is not that much greater than 16. We can also see that $\sigma = 4$ lies well inside the 80% confidence interval we computed earlier.
- For (ii), our hypotheses are $H_0 : \sigma^2 = 225$ and $H_a : \sigma^2 < 225$, since in fact the sample standard deviation is less than 225.
- Our test statistic is $\chi^2 = \dfrac{(n-1)S^2}{\sigma^2} = \dfrac{5 \cdot 4.6762^2}{225} = 0.4859$, and so the p-value is $P(Q_5 < 0.4859) = 0.00737$.
- Since the p-value is less than both significance levels, we fail reject the null hypothesis in both cases.
- This result is also reasonable, since the sample variance is quite a bit less than 225. We can also see that $\sigma = 15$ lies well outside the 80% confidence interval we computed earlier, but it is inside the 99% confidence interval (corresponding to the fact that the p-value is greater than 0.005).

5.2.3 The χ^2 Test For Goodness of Fit

- We often have reasons to believe that sample data should adhere to a particular shape or distribution. However, in many cases, we need to verify whether a particular model actually fits the data set we have collected.
  - In situations where we have a single variable of interest, we can often use the hypothesis tests we have already developed to test the reasonableness of a model.
  - For example, our z-test for unknown proportion is, very directly, testing whether a particular Bernoulli random variable is a good model for the observed data set (i.e., the collection of successes and failures observed in a sequence of Bernoulli trials).
  - However, most situations have a wider array of data values that we will want to compare to a prediction, and the hypothesis tests we have previously developed are not suitable for that more complicated task.
  - For example, we might want to test whether a die is fair by rolling it many times and tabulating the number of times each of the outcomes 1-6 is observed.
  - Of course, when we roll the die, we do not expect to get a proportion of precisely $1/6$ for each possible outcome (indeed, the distribution of the number of each roll will be binomially distributed).
  - What we want is a way to combine these results into a single test statistic to determine whether all of the results are collectively reasonable or unreasonable.
- The following theorem of Pearson gives a χ^2 test statistic for precisely this type of scenario where values are drawn from a discrete random variable:
- Theorem (χ^2 Goodness of Fit): Suppose that a discrete random variable E has outcomes $e_1, e_2, \ldots, e_k$ with respective probabilities $p_1, p_2, \ldots, p_k$. If we sample this random variable n times, obtaining the respective outcomes $e_1, e_2, \ldots, e_k$ a total of $x_1, x_2, \ldots, x_k$ times, then as $n \to \infty$ the random variable $D = \dfrac{(x_1 - np_1)^2}{np_1} + \dfrac{(x_2 - np_2)^2}{np_2} + \cdots + \dfrac{(x_k - np_k)^2}{np_k}$ is χ^2-distributed with $k-1$ degrees of freedom.
  - Note that each individual total $x_1, x_2, \ldots, x_k$ is binomially distributed (n trials, success probability p_i). The precise joint distribution of all of these totals is called a multinomial distribution.
  - Thus, the quantity np_i represents the expected number of times we would expect to see the outcome e_i if we sample the random variable n times.
  - As a practical matter, the approximation will be good whenever the expected frequencies np_i are all at least 5 or so.
  - We will not prove this theorem, as the actual details are quite technical (the idea relies on using moment-generating functions).
  - However, we can give some brief motivation: since x_i is binomially distributed, in the scenario where the normal approximation to the binomial is good, then x_i is approximately normally distributed with mean np_i and standard deviation $\sqrt{np_i(1-p_i)}$.

- Equivalently, that means $\dfrac{x_i - np_i}{\sqrt{np_i}}$ is approximately normally distributed with mean 0 and standard deviation $\sqrt{1-p_i}$, and so the quantity $(1-p_i)\dfrac{(x_1-np_1)^2}{np_1}$ is approximately χ^2-distributed with 1 degree of freedom.
- Summing over all of the random variables and noting that $(1-p_1)+(1-p_2)+\cdots+(1-p_n) = n-1$ shows that D is essentially the sum of $n-1$ χ^2-distributed variables each with 1 degree of freedom, which is equivalent to saying that it is a χ^2-distributed variable with $n-1$ degrees of freedom.
- This argument is not rigorous because it does not account for the non-independence of the totals; it is only intended as an approximate outline of the main ideas.

- Using this theorem, we can give a hypothesis testing procedure for analyzing the goodness of fit of a model:
  - We take our test statistic as $d = \dfrac{(x_1-np_1)^2}{np_1} + \dfrac{(x_2-np_2)^2}{np_2} + \cdots + \dfrac{(x_k-np_k)^2}{np_k} = \displaystyle\sum_{\text{data}} \frac{[\text{Observed} - \text{Expected}]^2}{\text{Expected}}$.
  - Our hypotheses are usually $H_0 : d = 0$ and $H_a : d > 0$, since the value $d = 0$ means the model is perfect and a positive value of d indicates deviation from the model.
  - In order to apply Pearson's result above, we must verify that most of the predicted observation sizes np_i are at least 5. Again, this is a heuristic estimate, so many different versions of a criterion are possible here. We will adopt the convention that at least 80% of the entries should be at least 5 or larger. Another option is to combine some of these small entries into groups that have a predicted size greater than 5.
  - If that is the case, then the test statistic is χ^2-distributed with $k-1$ degrees of freedom, and we can calculate the p-value as $P(Q_{k-1} \geq d)$.
  - We then compare the p-value to the significance level and then either reject or fail to reject the null hypothesis, as usual.

- Example: To test for fairness, a six-sided die is rolled 2000 times, yielding the results below. Test at the 10%, 3%, and 0.4% significance levels whether the die is fair.

Outcome	1	2	3	4	5	6
Observed	354	347	318	312	333	336

  - If the die is fair, we would expect each outcome to occur with probability 1/6, meaning that the expected totals are $2000/6 = 333.\overline{3}$ for each of the six possibilities.
  - Our test statistic is $d = \dfrac{(354-333.\overline{3})^2}{333.\overline{3}} + \dfrac{(347-333.\overline{3})^2}{333.\overline{3}} + \dfrac{(318-333.\overline{3})^2}{333.\overline{3}} + \dfrac{(312-333.\overline{3})^2}{333.\overline{3}} + \dfrac{(333-333.\overline{3})^2}{333.\overline{3}} + \dfrac{(336-333.\overline{3})^2}{333.\overline{3}} = 3.934$.
  - We can tabulate the test statistic a bit more conveniently by adding two extra rows to the table:

Outcome	1	2	3	4	5	6
Observed	354	347	318	312	333	336
Expected	$333.\overline{3}$	$333.\overline{3}$	$333.\overline{3}$	$333.\overline{3}$	$333.\overline{3}$	$333.\overline{3}$
$(O-E)^2/E$	$1.281\overline{3}$	$0.560\overline{3}$	$0.705\overline{3}$	$1.365\overline{3}$	$0.000\overline{3}$	$0.021\overline{3}$

  - Since there are 6 possible outcomes, there are $6-1 = 5$ degrees of freedom.
  - Thus, the p-value is $P(Q_5 \geq 3.934) = 0.5590$. Since this is well above each of our significance levels, we fail to reject the null hypothesis in each case.
  - Remark: The values were obtained by actually simulating a fair die roll, so it is not surprising that the p-value is large!

- Example: To determine whether a pollster is actually conducting their polls, the tenths-place digits from a random sample of 200 of their reported results are tabulated. The results are given below. It is expected that the tenths-place digit from poll percentages of thousands of people should be essentially uniformly distributed. Test at the 10%, 1%, and 0.02% significance levels whether the data appear to adhere to a uniform model.

Tenths Digit	0	1	2	3	4	5	6	7	8	9
Observed	7	26	13	44	25	10	9	41	12	13

- Here are the expected and χ^2-statistic values added to the table:

Tenths Digit	0	1	2	3	4	5	6	7	8	9
Observed	7	26	13	44	25	10	9	41	12	13
Expected	20	20	20	20	20	20	20	20	20	20
$(O-E)^2/E$	8.35	1.8	2.45	28.8	1.25	5	6.05	22.05	3.2	2.45

- Our test statistic is $d = 8.45 + 1.8 + 2.45 + 28.8 + 1.25 + 5 + 6.05 + 22.05 + 3.2 + 2.45 = 81.5$.
- There are 10 possible outcomes hence $10 - 1 = 9$ degrees of freedom.
- Thus, the p-value is $P(Q_9 \geq 81.5) = 8.13 \cdot 10^{-14}$. This is extremely small, so we reject the null hypothesis at all of the indicated significance levels.
- Remark: We can see here that the digits 3 and 7 were substantially overused, while 0 was underused. This sort of tendency to overuse certain digits and underuse others is common when humans try to generate lists of random digits.

- Example: It is believed that a Poisson model is appropriate to model the number of collisions at a particular busy intersection in a given week. The collisions are tabulated over a 5-year period (a total of 261 weeks), and the results are given below. Test at the 9% and 1% significance levels the accuracy of the model with parameter (i) $\lambda = 2.2$, and (ii) $\lambda = 2.9$.

# Collisions	0	1	2	3	4	5	6	7+
Observed	17	45	66	55	38	21	12	7

- If the Poisson model is accurate, we would expect the proportion of outcomes yielding d collisions to be $\frac{\lambda^d e^{-\lambda}}{d!}$, so the expected number of occurrences would be 261 times this quantity.
- For (i), here are the results for $\lambda = 2.2$ added to the table:

# Collisions	0	1	2	3	4	5	6	7+
Observed	17	45	66	55	38	21	12	7
Expected	28.92	63.63	69.99	51.32	28.23	12.42	4.55	1.95
$(O-E)^2/E$	4.9128	13.7927	0.2270	0.2635	3.3833	5.9271	12.1744	13.1090

- Here, we have 2 entries out of 8 that are less than 5. This is a sufficiently large percentage that we can use our χ^2 test.
- Our test statistic is $d = 4.9128+13.7927+0.2270+0.2635+3.3833+5.9271+12.1744+13.1090 = 53.7898$.
- Since there are 8 possible outcomes, there are $8 - 1 = 7$ degrees of freedom.
- Thus, the p-value is $P(Q_7 \geq 53.7898) = 2.588 \cdot 10^{-9}$. Since this is far below our significance levels, we reject the null hypothesis in both cases.
- For (ii), here are the results for $\lambda = 2.9$ added to the table:

# Collisions	0	1	2	3	4	5	6	7+
Observed	17	45	66	55	38	21	12	7
Expected	14.36	41.65	60.39	58.38	42.32	24.55	11.86	7.50
$(O-E)^2/E$	0.4849	0.2699	0.5215	0.1952	0.4414	0.5125	0.0016	0.0327

- Our test statistic is $d = 0.4849 + 0.2699 + 0.5215 + 0.1952 + 0.4414 + 0.5125 + 0.0016 + 0.0327 = 2.4597$.
- As above there are 7 degrees of freedom, so the p-value is $P(Q_7 \geq 2.4597) = 0.9301$. This is quite large, so we fail to reject the null hypothesis.
- Remark: The data set was generated by sampling a Poisson distribution whose actual parameter was $\lambda = 2.9$, so it is unsurprising that the null hypothesis is not rejected here!

- In this last example, we could have performed a maximum likelihood estimation for the Poisson parameter to find the ideal λ fitting the observed data.
  - The maximum likelihood estimator for that example ends up being $\hat{\lambda} = 2.7586$, which is not far from the actual value.
  - However, if we do this sort of "tuning" of the model to fit the data, we would expect to get somewhat better agreement than without being able to adjust a parameter to get a better fit.

- In order to correct for this, if we use a model with r unknown parameters that have been calculated to obtain optimal fit to the observed data, we should use a χ^2 test with $k-1-r$ degrees of freedom.
- Roughly speaking, each unknown parameter removes one degree of freedom from the hypothesis test, since each parameter value we are allowed to choose will allow us to model one additional outcome from the list of k correctly.

- We can also use a χ^2 test to decide whether a model appears "too good to believe", which would be the case if we are investigating data that may have been falsified or altered to adhere too closely to a model.
  - In such situations, we take the one-sided alternative hypothesis in the opposite direction: our null hypothesis is $H_0 : d = c$ with alternative hypothesis $H_a : d < c$ for (an arbitrary) positive c, and we would compute the p-value instead as $P(Q_{k-1} \leq d)$.

- Example: The pollster from the earlier example, in response to the accusations from the previous test, defends their innocence by sending another sample of tenths digits from 200 polls, which should again be uniformly distributed. Test at the 10%, 1%, and 0.02% significance levels whether the results are believable.

Tenths Digit	0	1	2	3	4	5	6	7	8	9
Observed	22	21	20	20	19	18	20	19	21	20

  - As before, we would expect with a perfect model to get an entry of 20 in each row.
  - This time, the numbers are all very suspiciously close to 20, so we will test whether the uniform model is too accurate (with the left tail for the distribution rather than the right tail).
  - Here are the expected and χ^2-statistic values added to the table:

Tenths Digit	0	1	2	3	4	5	6	7	8	9
Observed	22	21	20	20	19	18	20	19	21	20
Expected	20	20	20	20	20	20	20	20	20	20
$(O-E)^2/E$	0.2	0.05	0	0	0.05	0.2	0	0.05	0.05	0

  - Our test statistic is $d = 0.2 + 0.05 + 0 + 0 + 0.05 + 0.2 + 0 + 0.05 + 0.05 + 0 = 0.6$.
  - There are 10 possible outcomes hence $10 - 1 = 9$ degrees of freedom.
  - Thus, the p-value is $P(Q_9 \leq 0.6) = 6.64 \cdot 10^{-5}$. This is extremely small, so we reject the null hypothesis at all of the indicated significance levels.
  - Our interpretation of the hypothesis test is that these observed values fit the model too closely to be random, and thus it seems that the results are too good to be true, and thus are likely forged or manipulated somehow.

5.2.4 The χ^2 Test for Independence

- As a final application of the χ^2 test, we will apply it to study the independence of discrete random variables.
  - Recall that we can test whether two discrete random variables X and Y are independent by checking whether $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$.
  - If we construct a joint probability distribution table, we can check whether X and Y are independent by computing the row and column sums, and then testing whether each entry $p_{X,Y}(x, y)$ in the table is the product of its associated row sum $p_X(x)$ and its associated column sum $p_Y(y)$.
  - Now suppose we are computing the joint distribution table for two random variables X and Y by sampling a population. We would expect the entries in the resulting table (which are now counts of individual observations) to show some random variation in their values away from the true proportion $p_{X,Y}(x, y)$.
  - Thus, if we try to determine whether X and Y are independent using the criterion $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$, it is very unlikely that we would see exact independence.
  - We can, however, adapt Pearson's χ^2 test for goodness-of-fit to give a hypothesis test for independence: the scenario we are describing is essentially identical to the one we just analyzed.

- Theorem (χ^2 Independence): Suppose that the discrete random variables X and Y have outcomes $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$. Suppose that (X, Y) is sampled n times, such that the outcome x_i occurs a proportion p_i times, the outcome y_j occurs a proportion q_j times, and the outcome pair (x_i, y_j) occurs $a_{i,j}$ times for each $1 \leq i \leq a$ and $1 \leq j \leq b$. Then, as $n \to \infty$, the random variable $D = \displaystyle\sum_{i=1}^{a} \sum_{j=1}^{b} \frac{(a_{i,j} - np_iq_j)^2}{np_iq_j}$ is χ^2-distributed with $(a-1)(b-1)$ degrees of freedom.
  - The idea is that if X and Y are independent, then np_iq_j is the expected number of times we should obtain the outcomes x_i (probability p_i) and y_j (probability q_j) together.
  - Thus, we are computing the same sum $D = \displaystyle\sum_{\text{data}} \frac{[\text{Observed} - \text{Expected}]^2}{\text{Expected}}$ as before.
  - The proof of this result is similar to the one we gave earlier for goodness-of-fit: for large n, each of the ratios $\dfrac{(a_{i,j} - np_iq_j)^2}{np_iq_j}$ will behave like a scaled χ^2 distribution with 1 degree of freedom.
- We will briefly explain the non-obvious fact about why the number of degrees of freedom is $(a-1)(b-1)$.
  - Essentially, the idea is that if we are filling entries into the joint pdf table of X and Y, then all of the entries in the $a \times b$ table are completely determined once we fill in the upper left $(a-1) \times (b-1)$ table, under the presumption that we also know the row and column sums p_i and q_j (because we extract p_i and q_j from the data, we view them as parameters that we have selected).
  - We can fill in all the entries because once we have all but one entry in a given row, we can fill in the last entry since we know the row sum. The same holds true for the columns, so applying this for each row and column (including the bottom row that we just filled) allows us to fill the entire grid.
  - On the other hand, if we have fewer than $(a-1)(b-1)$ entries, we cannot fill the entire grid. Thus, the total number of independent values is $(a-1)(b-1)$, so this is the number of degrees of freedom.
  - An equivalent (and more highbrow) way to make this observation is that the entries in the upper $(a-1) \times (b-1)$ subgrid form a basis for the vector space consisting of the entries of the grid with fixed row and column sums.
- Using this theorem, we can give a hypothesis testing procedure for analyzing the independence of two random variables X and Y:
  - First, we write down the $a \times b$ joint probability distribution table for the observed values of X and Y, and compute the row proportions p_i and column proportions q_j.
  - Then we compute the expected value of each entry np_iq_j, and calculate the test statistic as $d = \displaystyle\sum_{i=1}^{a} \sum_{j=1}^{b} \frac{(a_{i,j} - np_iq_j)^2}{np_iq_j} = \sum_{\text{data}} \frac{[\text{Observed} - \text{Expected}]^2}{\text{Expected}}$.
  - We take as our hypotheses $H_0 : d = 0$ and $H_a : d > 0$, since the value $d = 0$ means that the model is perfect (indicating that all of the entries are exactly equal to the predicted value, which means X and Y are independent) and a positive value of d indicates deviation from independence.
  - In order to apply Pearson's result above, we must verify that most of the predicted observation sizes np_i are at least 5. We will adopt the same convention as above, that at least 80% of the entries should be at least 5 or larger.
  - If that is the case, then the test statistic is χ^2-distributed with $(a-1)(b-1)$ degrees of freedom, and we can calculate the p-value as $P(Q_{(a-1)(b-1)} \geq d)$.
  - We then compare the p-value to the significance level and then either reject or fail to reject the null hypothesis, as usual.
- Example: The faculty members in a university mathematics department are either tenure-track or non-tenure-track. These categories are broken down further by gender as indicated below. Test at the 9% and 0.8% significance levels whether the two variables of tenure track status and gender are independent.

Observed	Tenure-Track	Non-Tenure-Track
Male	20	8
Female	4	8

- There are 40 faculty in total, so we can compute the row and column proportions and then fill in the table of expected values as follows:

Expected	Tenure-Track	Non-Tenure-Track	Proportion
Male	$40 \cdot 0.42 = 16.8$	$40 \cdot 0.28 = 11.2$	0.7
Female	$40 \cdot 0.18 = 7.2$	$40 \cdot 0.12 = 4.8$	0.3
Proportion	0.6	0.4	

- Then the test statistic is given by $\frac{(20-16.8)^2}{16.8} + \frac{(8-11.2)^2}{11.2} + \frac{(4-7.2)^2}{7.2} + \frac{(8-4.8)^2}{4.8} = 5.0794$.
- The total number of degrees of freedom is $(2-1)(2-1) = 1$, so the p-value is given by $P(Q_1 \geq 5.0794) = 0.02421$.
- Since the p-value is below the 9% significance level but above the 0.8% significance level, we reject the null hypothesis in the first case but not in the second case.
- Our interpretation of the test is that we have moderately strong evidence that the variables are not independent.

- Example: A survey is taken of 400 households asking about the number of children and the number of TVs in the household. Test at the 11% and 2% significance levels whether the number of TVs is independent of the number of children.

Observed	0 Children	1 Child	2 Children	3+ Children
0 TVs	10	25	29	16
1 TV	19	88	104	29
2+ TVs	9	24	29	18

- We compute the row and column proportions and then fill in the table of expected values as follows:

Expected	0 Children	1 Child	2 Children	3+ Children	Proportion
0 TVs	7.6	27.4	32.4	12.6	0.2
1 TV	22.8	82.2	97.2	37.8	0.6
2+ TVs	7.6	27.4	32.4	12.6	0.2
Proportion	0.095	0.3425	0.405	0.1575	

- Then the test statistic is given by $\frac{(10-7.6)^2}{7.6} + \frac{(25-27.4)^2}{27.4} + \cdots + \frac{(18-12.6)^2}{12.6} = 9.1602$.
- The total number of degrees of freedom is $(4-1)(3-1) = 6$, so the p-value is given by $P(Q_6 \geq 9.1602) = 0.1648$.
- Since the p-value is above the 11% and 2% significance levels, we fail reject the null hypothesis in both cases
- Our interpretation is that we have fairly weak evidence that the variables are not independent: the number of TVs and the number of children do not appear to be far off independence.

- Example: A poll is taken on a trenchant political issue and the support is broken down by age group, as shown below. Test at the 8%, 2%, and 0.3% significance levels whether the level of support is independent of the age group.

Observed	Age 18-29	Age 30-49	Age 50-64	Age 65+
Support	20	13	12	8
Oppose	7	9	14	17

- There are 100 responses in total, so we can compute the row and column proportions and then fill in the table of expected values as follows:

Expected	Age 18-29	Age 30-49	Age 50-64	Age 65+	Proportion
Support	14.31	11.66	13.78	13.25	0.53
Oppose	12.69	10.34	12.22	11.75	0.47
Proportion	0.27	0.22	0.26	0.25	

- Then the test statistic is given by $\frac{(20-14.31)^2}{14.31} + \frac{(13-11.66)^2}{11.66} + \cdots + \frac{(17-11.75)^2}{11.75} = 10.057$.

- The total number of degrees of freedom is $(4-1)(2-1) = 3$, so the p-value is given by $P(Q_3 \geq 10.057) = 0.01809$.
- Since the p-value is below the 8% and 2% significance levels, we reject the null hypothesis in those cases. However, it is above the 0.3% significance level, so we fail to reject the null hypothesis in that case.
- Our interpretation of the test is that we have fairly strong evidence that the variables are not independent: the support does appear to depend on the age group.

- We will remark that for 2×2 tables (i.e., the situation of 1 degree of freedom), there does exist an exact test due to Fisher, known as Fisher's exact test, that allows for performing a hypothesis test associated to a given table without the need for using a χ^2 approximation.
  - The idea is that if the row and column totals are known, then (as we have noted above) only the single upper-left entry is required to determine the full table.
  - Fisher's original example was "the lady tasting tea", who claimed to be able to decide, solely by the flavor, whether a cup of tea with milk had the milk poured into the tea or the tea poured into the milk.
  - Eight cups were poured, four with milk first and four with tea first; the lady tasted each and decided whether the tea or the milk had been poured first. Suppose that the results were as follows:

Observed	Lady: Milk first	Lady: Tea first
Milk poured first	a	b
Tea poured first	c	d

  - Under the null hypothesis of random guessing, we assume that the lady would guess exactly 4 cups of each type, since she was aware that there were 4 of each type.
  - Thus, to obtain the table above the lady will always guess $a+c$ of the cups to have milk first and $b+d$ to have tea first, so there are a total $\binom{a+b+c+d}{a+c}$ possible tables satisfying this condition.
  - To obtain the specific table above, exactly a of the $a+c$ cups the lady says have milk must actually have milk, and exactly d of the cups the lady says have tea must actually have tea. There are $\binom{a+c}{a} \cdot \binom{b+d}{d}$ ways of making these selections, so the total probability of obtaining the given table is $\binom{a+c}{a}\binom{b+d}{d} / \binom{a+b+c+d}{a+c}$.
  - We can then compute the probability of obtaining a result at least as extreme (in the direction of accuracy) by summing over the possible tables with upper-left entry at least as large as the observed value.
  - For example, if the results had been

Observed	Lady: Milk first	Lady: Tea first
Milk poured first	3	1
Tea poured first	1	3

    then the probability of obtaining this precise table is $\binom{4}{3}\binom{4}{3}/\binom{8}{4} = 16/70 \approx 0.2286$. The only result yielding more correct responses would be the table with entries $(4,0),(0,4)$ which occurs with probability $\binom{4}{4}\binom{4}{4}/\binom{8}{4} = 1/70 \approx 0.0143$. Thus, the tail probability is the sum $16/70 + 1/70 \approx 0.2429$. We would likely not view this as conclusive evidence.
  - In fact, the results of the actual test were that the lady correctly identified all 8 cups. In that case, the probability of obtaining the result by random guessing is $\binom{4}{4}\binom{4}{4}/\binom{8}{4} = 1/70 \approx 0.0143$: much more compelling!
  - As a final remark, we will note that the book in which Fisher described this example marked the first appearance of the notion of testing a null hypothesis by calculating a p-value and comparing it to a significance threshold.

Well, you're at the end of my handout. Hope it was helpful.