# Math 3081 (Probability and Statistics)

## Lecture #10 of 27 $\sim$ July 21st, 2021

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Continuous Random Variables (Part 2)

- More with Chebyshev's and Markov's Inequalities
- Double Integrals
- Joint Distributions

This material represents §2.2.2-2.2.3 from the course notes, and problems 15-18 on WeBWorK 3 + problem 20 on WeBWorK 4.

# Recall, I

Last time, we showed some properties of expected value:

**Proposition (Expected Value of Functions of $X$)**

*If $X$ is a continuous random variable with probability density function $p(x)$, and $g(x)$ is any piecewise-continuous function, then the expected value of $g(X)$ is $E[g(X)] = \int_{-\infty}^{\infty} g(x)\, p(x)\, dx$.*

**Corollary (Linearity of Expected Value)**

*If $X$ and $Y$ are continuous random variables whose expected values are defined, and $a$ and $b$ are any real numbers, then $E(aX + b) = a \cdot E(X) + b$ and $E(X + Y) = E(X) + E(Y)$.*

# Recall, II

We also defined variance and standard deviation:

**Definition**

*If $X$ is a continuous random variable whose expected value $\mu$ exists and is finite, the variance $\mathrm{var}(X)$ is defined as $\mathrm{var}(X) = E[(X-\mu)^2] = E(X^2) - E(X)^2$, and the standard deviation is $\sigma(X) = \sqrt{\mathrm{var}(X)}$.*

And we showed a few properties:

**Proposition (Properties of Variance)**

*If $X$ is a continuous random variable and $a$ and $b$ are any real numbers, then $\mathrm{var}(aX+b) = a^2 \cdot \mathrm{var}(X)$ and $\sigma(aX+b) = |a|\,\sigma(X)$.*

# Recall, III

We also mentioned Markov's + Chebyshev's inequalities:

**Theorem (Markov's Inequality)**

*If $Y$ is a nonnegative random variable and $a$ is any positive real number, then $P(Y \geq a) \leq E(Y)/a$.*

**Theorem (Chebyshev's Inequality)**

*If $X$ is a random variable with expected value $\mu$ and standard deviation $\sigma$, then $P(|X - \mu| \geq k\sigma) \leq 1/k^2$ for any positive real number $k$.*

Both of these inequalities hold for any random variable, discrete or continuous. They are very general statements.

# More Chebyshev and Markov, I

Example: In a statistics class, a student's exam score is a random variable with mean 80, but no other information about the exam scores is known.

1. Find an upper bound for the proportion of students who receive a 90 or above.
2. If the standard deviation is also known to be 6, find a lower bound for the proportion of students who score between 70 and 90 (inclusive).
3. If the probability that the class average is within 1 point of the mean is at least 84%, without using any information about the distribution other than the mean and standard deviation, what is the least number of students that could be in the class?

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- The only tools here are Markov's + Chebyshev's inequalities.

# More Chebyshev and Markov, II

Example: In a statistics class, a student's exam score is a random variable with mean 80, but no other information about the exam scores is known.

1. Find an upper bound for the proportion of students who receive a 90 or above.

- Markov's inequality: if $Y$ is nonnegative and $a > 0$, then $P(Y \geq a) \leq E(Y)/a$.

# More Chebyshev and Markov, II

<u>Example</u>: In a statistics class, a student's exam score is a random variable with mean 80, but no other information about the exam scores is known.

1. Find an upper bound for the proportion of students who receive a 90 or above.

- Markov's inequality: if $Y$ is nonnegative and $a > 0$, then $P(Y \geq a) \leq E(Y)/a$.
- We can use the inequality here (at least, under the sensible assumption that the exam scores are always nonnegative).
- If $Y$ is the exam score and $a = 90$, then we have $P(Y \geq a) \leq E(Y)/90 = 80/90$.
- Thus, at most $8/9$ of the class could score 90 or above.
- Try to convince yourself that this is the most we can say without any other information.

# More Chebyshev and Markov, III

Example: In a statistics class, a student's exam score is a random variable with mean 80, but no other information about the exam scores is known.

2. If the standard deviation is also known to be 6, find a lower bound for the proportion of students who score between 70 and 90 (inclusive).

- Chebyshev's inequality: a proportion at least $1 - 1/k^2$ of students must score within $k$ standard deviations of the mean.

# More Chebyshev and Markov, III

Example: In a statistics class, a student's exam score is a random variable with mean 80, but no other information about the exam scores is known.

2. If the standard deviation is also known to be 6, find a lower bound for the proportion of students who score between 70 and 90 (inclusive).

- Chebyshev's inequality: a proportion at least $1 - 1/k^2$ of students must score within $k$ standard deviations of the mean.
- We are given the standard deviation, so we can apply the result here.
- Since the given score range of $80 \pm 10$ points represents $k = 10/6$ standard deviations away from the mean, Chebyshev's inequality says that the required proportion of students is at least $1 - 1/k^2 = 0.64$.
- Thus, at least 64% of students must score between 70 and 90.

# More Chebyshev and Markov, IV

<u>Example</u>: In a statistics class, a student's exam score is a random variable with mean 80 and standard deviation 6.

3. If the probability that the class average is within 1 point of the mean is at least 84%, without using any other information, what is the least number of students that could be in the class?

# More Chebyshev and Markov, IV

Example: In a statistics class, a student's exam score is a random variable with mean 80 and standard deviation 6.

3. If the probability that the class average is within 1 point of the mean is at least 84%, without using any other information, what is the least number of students that could be in the class?

- We don't know what the distribution of the average score looks like, so the only tool is Chebyshev's inequality.
- If there are $n$ students in the class, then the average score has mean 80 and standard deviation $6/\sqrt{n}$.
- Chebyshev's inequality says that the proportion of students that score within $k$ standard deviations of the mean is at least $1 - 1/k^2$, which is equal to 84% for $k = 5/2$.
- Thus, 1 point must represent $5/2$ of a standard deviation $6/\sqrt{n}$: this means $1 = (5/2) \cdot (6/\sqrt{n})$ so that $n = 225$.

# Joint Distributions, I

Next, we discuss the situation of having several continuous random variables defined on the same sample space.

- Just as with discrete random variables, if we have a collection of continuous random variables $X_1, X_2, \ldots, X_n$, we can summarize all of the possible information about the behavior of these random variables simultaneously using a joint probability density function.
- In the discrete case, we can find the probability that the random variables take particular combinations of values by summing the associated values of the joint distribution function.
- In the continuous case, we will instead need to integrate to find these probabilities.

# Joint Distributions, II

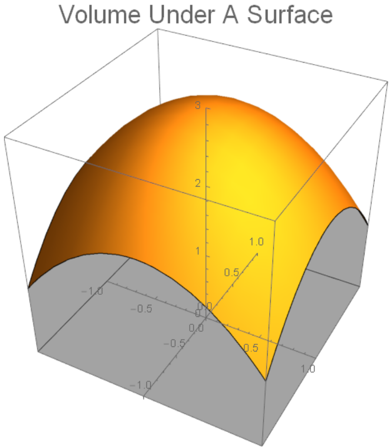
**Definition**

*If $X$ and $Y$ are continuous random variables, and there is a function $p_{X,Y}(x,y)$, defined on ordered pairs of real numbers $(x,y)$ such that $\iint_R p_{X,Y}(x,y)\,dy\,dx = P[(X,Y) \in R]$ for every plane region $R$, then we call $p_{X,Y}(x,y)$ the joint probability density function of $X$ and $Y$.*

- The integral in the definition above is a double integral over the region $R$.
- The idea is analogous to a one-variable probability density function: to compute the probability that the values of of $X$ and $Y$ together land in a particular planar region $R$, we simply integrate the probability density function on the domain $R$.

# Double Integrals, I

We can interpret the double integral $\iint_R p(x, y)\, dy\, dx$ as a volume: specifically, it represents the volume underneath the surface $z = p(x, y)$ that lies above the planar region $R$:



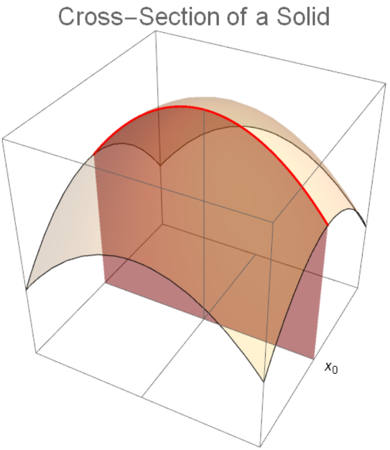
# Double Integrals, II

To evaluate probabilities using a joint density function, we will need to use multivariable integration to evaluate $\iint_R p_{X,Y}(x, y)\, dy\, dx$.

- Abstractly, this double integral is defined in terms of Riemann sums. We will not work with this definition: instead, we will evaluate $R$ as what is called an iterated integral.
- To motivate the idea, first suppose the region $R$ is the rectangle with $a \le x \le b$, $c \le y \le d$, usually written as $[a, b] \times [c, d]$ for short.
- Imagine taking the solid volume and slicing it into thin pieces perpendicular to the $x$-axis from $x = a$ to $x = b$, then the volume is given by the integral $\int_a^b A(x)\, dx$, where $A(x)$ is the cross-sectional area at a given $x$-coordinate.

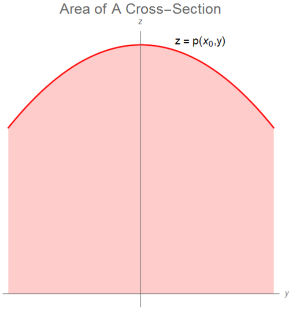
# Double Integrals, III

Here is a picture of a typical cross-section at the $x$-coordinate $x_0$:



# Double Integrals, IV

Now look at each cross-section:




- Notice that the area $A(x_0)$ is simply the area under the curve $z = p(x_0, y)$ between $y = c$ and $y = d$.
- That area is $\int_c^d p(x_0, y)\, dy$, where here we are thinking of $x_0$ as a constant independent of $y$.

# Double Integrals, V

Putting this together shows that the volume under $z = p(x, y)$ above the region $R = [a, b] \times [c, d]$ is given by the iterated integral

$$\int_a^b \left[ \int_c^d p(x, y)\, dy \right] dx.$$

- In this integral, we integrate first (on the inside) with respect to the variable $y$, and then second (on the outside) with respect to the variable $x$.
- We will usually write iterated integrals without the brackets: $\int_a^b \int_c^d p(x, y)\, dy\, dx$.
- Note that there are two limits of integration, and they are paired with the two variables "inside out": the inner limits $[c, d]$ are paired with the inner differential $dy$, and the outer limits $[a, b]$ are paired with the outer differential $dx$.

## Double Integrals, VI

Example: Find the volume under the surface $z = x + y$ that lies above the region $0 \leq x \leq 1$, $0 \leq y \leq 2$.

# Double Integrals, VI

Example: Find the volume under the surface $z = x + y$ that lies above the region $0 \le x \le 1$, $0 \le y \le 2$.

- The volume is given by the iterated integral $\int_0^1 \int_0^2 (x+y)\,dy\,dx$.
- To evaluate the inner integral $\int_0^2 (x+y)\,dy$, we view $x$ as a constant and take the antiderivative (with respect to $y$):

$$\begin{aligned}\int_0^2 (x+y)\,dy &= \left(xy + \frac{1}{2}y^2\right)\Big|_{y=0}^{2} \\ &= (2x+2) - (0+0) = 2x+2.\end{aligned}$$

- Now we can plug this in and evaluate the outer integral:

$$\int_0^1 \int_0^2 (x+y)\,dy\,dx = \int_0^1 (2x+2)\,dx = (x^2+2x)\Big|_{x=0}^{1} = 3.$$

# Double Integrals, VII

<u>Example</u>: Calculate $\int_0^2 \int_0^3 x^2 y \, dy \, dx$.

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Example: Calculate $\int_0^2 \int_0^3 x^2 y \, dy \, dx$.

- We evaluate the integrals, starting with the inner integral.
- To evaluate the inner integral, we take the antiderivative of $x^2 y$ with respect to $y$, and then evaluate from $y = 0$ to $y = 3$. This gives us a function just of $x$, which we then integrate from $x = 0$ to $x = 2$:

$$\begin{aligned} \int_0^2 \int_0^3 x^2 y \, dy \, dx &= \int_0^2 \left[ \int_0^3 x^2 y \, dy \right] dx = \int_0^2 \left[ \frac{1}{2} x^2 y^2 \right] \Big|_{y=0}^{3} dx \\ &= \int_0^2 \frac{9}{2} x^2 \, dx = \frac{3}{2} x^3 \Big|_{x=0}^{2} = 12. \end{aligned}$$

# Double Integrals, VIII

We can also deal with the more general situation where the region $R$ is not a rectangle.

- There are a number of more general approaches that work for various situations (e.g., if the region is a disc, one could use polar coordinates).
- We only deal with the case where $R$ is bounded below by the curve $y = c(x)$ and above by the curve $y = d(x)$.
- By thinking again in terms of cross-sections, the iterated integral will have the form $\int_a^b \int_{c(x)}^{d(x)} p(x, y)\, dy\, dx$, where now the inner limits of integration depend on the outer variable $x$.
- When we evaluate the inner integral in $y$, we will be left with a function of $x$, and then we can evaluate the outer integral.

# Double Integrals, IX

Example: Calculate $\int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx$.

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Example: Calculate $\int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx$.

- We evaluate the integrals, starting with the inner integral, just like before.
- When we compute the inner integral, the limits will be in terms of $x$ (just plug them in like normal). The result is then a function of $x$, at which point we evaluate the outer integral:

$$\begin{aligned} \int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx &= \int_0^2 \frac{1}{3}xy^3 \Big|_{y=x^2}^{2x} \, dx = \int_0^2 \left[ \frac{8}{3}x^4 - \frac{1}{3}x^7 \right] dx \\ &= \left[ \frac{8}{15}x^5 - \frac{1}{24}x^8 \right] \Big|_{x=0}^{2} = \frac{32}{5}. \end{aligned}$$

# Double Integrals, X

In most situations we will need to set up the double integral of a function $p(x, y)$ on a region $R$ using only a description of $R$.

- The procedure is as follows: first, sketch the region.
- Next, "slice up" the region using vertical cross-sections.
- Identify the value $x = a$ where the leftmost cross-section is, and the value $x = b$ where the rightmost cross-section is.
- Inside any given cross-section, identify the curve $y = c(x)$ representing the bottom boundary, and the curve $y = d(x)$ representing the upper boundary.
- The desired double integral is then $\int_a^b \int_{c(x)}^{d(x)} p(x, y)\, dy\, dx$.

# Double Integrals, XI

Example: Set up the integral of $p(x, y) = xy$ on the region with $x, y \geq 0$ and $x + y \leq 1$.

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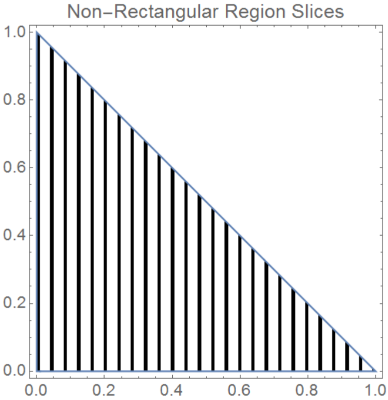
<u>Example</u>: Set up the integral of $p(x,y) = xy$ on the region with $x, y \geq 0$ and $x + y \leq 1$.

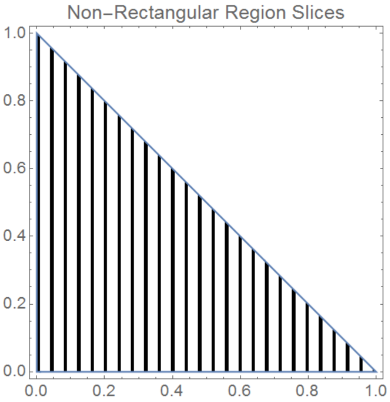
- First, we sketch the region, and then we slice it up vertically:



## Double Integrals, XI

Example: Set up the integral of $p(x, y) = xy$ on the region with $x, y \geq 0$ and $x + y \leq 1$.

- First, we sketch the region, and then we slice it up vertically:




- The leftmost slice occurs at $x = 0$ and the rightmost slice occurs at $x = 1$.
- For each slice, the bottom curve is $y = 0$ and the upper curve is the line $x + y = 1$, which (since we need $y$ as a function of $x$) is $y = 1 - x$.
- Therefore, the desired double integral is $\int_0^1 \int_0^{1-x} xy \, dy \, dx$.

# What Were We Doing Again? Oh, Right, Probability....

Now that we have done the necessary basics for double integrals, we can get back to talking about joint distributions.

- The main takeaway is that if $p_{X,Y}(x, y)$ is the joint distribution for $X$ and $Y$, then we can calculate probabilities of joint events by evaluating the double integral of $p_{X,Y}$ on the appropriate region.
- The only hard part is setting up the double integral. (And then evaluating it.)

# Joint Distribution Examples, I

<u>Example</u>: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = \frac{1}{4}xy$ for $0 \le x \le 2$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

1. Verify that $p$ is a joint probability density function.
2. Find $P(0 \le X \le 1 \text{ and } 0 \le Y \le 1)$.
3. Find $P(0 \le X \le 1)$.

# Joint Distribution Examples, I

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- We just need to set up the appropriate integrals.
- We can restrict our attention to the rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 2$ since that is the only place where the density function is nonzero.

# Joint Distribution Examples, II

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = \frac{1}{4}xy$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

1. Verify that $p$ is a joint probability density function.

# Joint Distribution Examples, II

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = \frac{1}{4}xy$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

1. Verify that $p$ is a joint probability density function.

- For this, we need to verify that the integral of $p$ is equal to 1.
- Since the region of interest is a rectangle, the integral is

$$\begin{aligned} \int_0^2 \int_0^2 \frac{1}{4} xy \, dy \, dx &= \int_0^2 \frac{1}{8} xy^2 \Big|_{y=0}^{2} dx \\ &= \int_0^2 \frac{1}{2} x \, dx = \frac{1}{4} x^2 \Big|_{x=0}^{2} = 1. \end{aligned}$$

# Joint Distribution Examples, III

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = \frac{1}{4}xy$ for $0 \le x \le 2$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

2. Find $P(0 \le X \le 1 \text{ and } 0 \le Y \le 1)$.

# Joint Distribution Examples, III

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = \frac{1}{4}xy$ for $0 \le x \le 2$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

2. Find $P(0 \le X \le 1 \text{ and } 0 \le Y \le 1)$.

- The region of interest is now the portion where $0 \le x \le 1$ and $0 \le y \le 1$.
- The integral is

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{1}{4}xy\,dy\,dx &= \int_0^1 \left[\frac{1}{8}xy^2\right]\Big|_{y=0}^{1} dx \\
&= \int_0^1 \frac{1}{8}x\,dx = \left[\frac{1}{16}x^2\right]\Big|_{x=0}^{1} = \frac{1}{16}.
\end{aligned}$$

# Joint Distribution Examples, IV

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = \frac{1}{4}xy$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

3. Find $P(0 \leq X \leq 1)$.

# Joint Distribution Examples, IV

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = \frac{1}{4}xy$ for $0 \le x \le 2$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

3. Find $P(0 \le X \le 1)$.

- The region of interest is now the portion where $0 \le x \le 1$ and $0 \le y \le 2$.
- The integral is

$$\begin{aligned}\int_0^1 \int_0^2 \frac{1}{4}xy \, dy \, dx &= \int_0^1 \left[\frac{1}{8}xy^2\right]\Big|_{y=0}^{2} dx \\ &= \int_0^1 \frac{1}{2}x \, dx = \left[\frac{1}{4}x^2\right]\Big|_{x=0}^{1} = \frac{1}{4}.\end{aligned}$$

# Joint Distribution Examples, V

<u>Example</u>: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = c(2x + y)$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

1. Find the value of $c$.
2. Find $P(0 \le X \le 1 \text{ and } 1 \le Y \le 2)$.
3. Find $P(0 \le Y \le 1)$.
4. Find $P(Y > X)$.
5. Find $P(X > 3Y)$.
6. Find $P(X > 2Y \text{ and } Y > X/3)$.

# Joint Distribution Examples, V

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = c(2x + y)$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

1. Find the value of $c$.
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3. Find $P(0 \leq Y \leq 1)$.
4. Find $P(Y > X)$.
5. Find $P(X > 3Y)$.
6. Find $P(X > 2Y \text{ and } Y > X/3)$.

- We just need to set up the appropriate integrals.
- We can restrict our attention to the rectangle where $0 \leq x \leq 3$ and $0 \leq y \leq 2$ since that is the only place where the density function is nonzero.

# Joint Distribution Examples, VI

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = c(2x + y)$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

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# Joint Distribution Examples, VI

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = c(2x + y)$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

1. Find the value of $c$.

- Since $p$ is a probability density function, its integral over its entire domain must equal zero.
- Ignoring the places where $p$ is zero gives the following integral:

$$\begin{aligned}\int_0^3 \int_0^2 c(2x+y)\,dy\,dx &= \int_0^3 c(2xy + y^2/2)\Big|_{y=0}^2\,dx \\ &= \int_0^3 c(4x+2)\,dx \\ &= c(2x^2+2x)\Big|_{x=0}^3 = 24c.\end{aligned}$$

- Therefore, we must have $24c = 1$ so $c = 1/24$.

# Joint Distribution Examples, VII

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

2. Find $P(0 \leq X \leq 1 \text{ and } 1 \leq Y \leq 2)$.

# Joint Distribution Examples, VII

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x,y) = (2x+y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x,y) = 0$ elsewhere. Do:

2. Find $P(0 \le X \le 1 \text{ and } 1 \le Y \le 2)$.

- Here, the region of interest has $0 \le x \le 1$ and $1 \le y \le 2$.
- This yields the following integral:

$$\begin{aligned}
\int_0^1 \int_1^2 \frac{1}{24}(2x+y)\,dy\,dx &= \int_0^1 \frac{1}{24}(2xy + y^2/2)\Big|_{y=1}^{2}\,dx \\
&= \int_0^1 \frac{1}{48}(4x+3)\,dx \\
&= \frac{1}{48}(2x^2+3x)\Big|_{x=0}^{1} = \frac{5}{48}.
\end{aligned}$$

# Joint Distribution Examples, VIII

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

3. Find $P(0 \leq Y \leq 1)$.

# Joint Distribution Examples, VIII

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

3. Find $P(0 \leq Y \leq 1)$.

- Here, there is no restriction on the value of $X$, so the region of interest has $0 \leq x \leq 3$ and $0 \leq y \leq 1$.
- This yields the following integral:

$$\begin{aligned} \int_0^3 \int_0^1 \frac{1}{24}(2x + y)\, dy\, dx &= \int_0^3 \frac{1}{24}(2xy + y^2/2)\Big|_{y=0}^1\, dx \\ &= \int_0^3 \frac{1}{48}(4x + 1)\, dx \\ &= \frac{1}{48}(2x^2 + x)\Big|_{x=0}^3 = \frac{7}{16}. \end{aligned}$$

# Joint Distribution Examples, IX

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

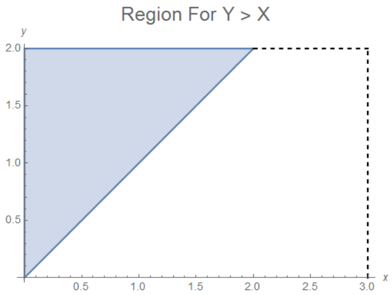
3. Find $P(Y > X)$.

# Joint Distribution Examples, IX

<u>Example</u>: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

3. Find $P(Y > X)$.

- The region is below.



# Joint Distribution Examples, IX

<u>Example</u>: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

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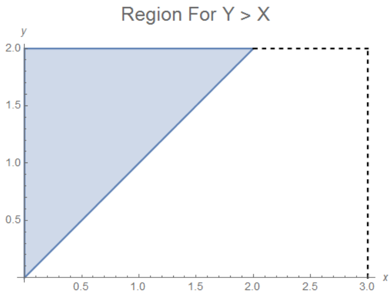



- Slicing it up gives the leftmost slice at $x = 0$ and the rightmost slice at $x = 2$.
- For each slice, the bottom curve is $y = x$ and the upper curve is the line $y = 2$.
- Therefore, the integral is $\int_0^2 \int_x^2 \frac{1}{24}(2x + y)\, dy\, dx$.

## Joint Distribution Examples, X: Are We There Yet?

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

3. Find $P(Y > X)$.

- Now we just have to evaluate $\int_0^2 \int_x^2 \frac{1}{24}(2x + y)\, dy\, dx$.
- This yields

$$\begin{aligned}
\int_0^2 \int_x^2 \frac{1}{24}(2x + y)\, dy\, dx &= \int_0^2 \frac{1}{24}(2xy + y^2/2)\Big|_{y=x}^2\, dx \\
&= \int_0^2 \frac{1}{48}(4 + 8x - 5x^2)\, dx \\
&= \frac{1}{144}(12x + 12x^2 - 5x^3)\Big|_{x=0}^2 = \frac{1}{16}.
\end{aligned}$$

# Joint Distribution Examples, XI: Nope

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

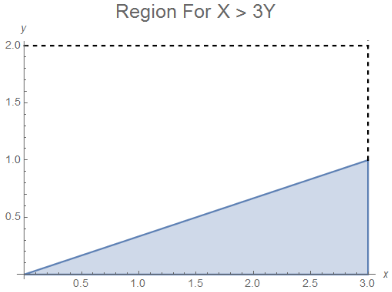
5. Find $P(X > 3Y)$.

# Joint Distribution Examples, XI: Nope

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

5. Find $P(X > 3Y)$.

- The region is below.

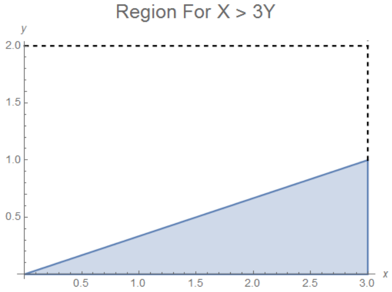


# Joint Distribution Examples, XI: Nope

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

5. Find $P(X > 3Y)$.

- The region is below.




- Slicing it up gives the leftmost slice at $x = 0$ and the rightmost slice at $x = 3$.
- For each slice, the bottom curve is $y = 0$ and the upper curve is the line $x = 3y$, which is $y = x/3$.
- Therefore, the integral is $\int_0^3 \int_0^{x/3} \frac{1}{24}(2x + y)\, dy\, dx$.

# Joint Distribution Examples, XII: Wait, There's More

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

5. Find $P(X > 3Y)$.

- Now we just have to evaluate $\displaystyle\int_0^3 \int_0^{x/3} \frac{1}{24}(2x + y)\, dy\, dx$.
- This yields

$$\begin{aligned}
\int_0^3 \int_0^{x/3} \frac{1}{24}(2x + y)\, dy\, dx &= \int_0^3 \frac{1}{24}(2xy + y^2/2)\Big|_{y=0}^{x/3} dx \\
&= \int_0^3 \frac{13}{432} x^2\, dx \\
&= \frac{13}{1296} x^3\Big|_{x=0}^{3} = \frac{13}{48}.
\end{aligned}$$

# Joint Distribution Examples, XIII: Unlucky

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere. Do:

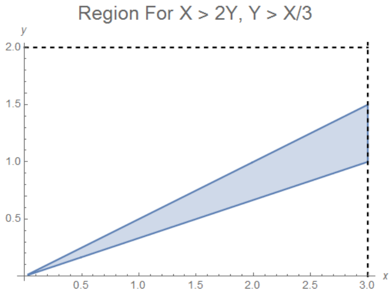
6. Find $P(X > 2Y \text{ and } Y > X/3)$.

# Joint Distribution Examples, XIII: Unlucky

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

6. Find $P(X > 2Y \text{ and } Y > X/3)$.

- The region is below.

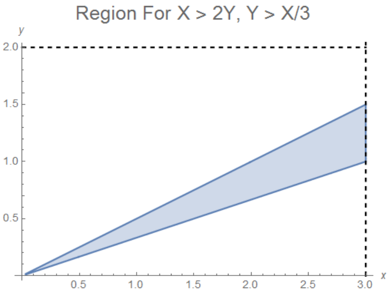


# Joint Distribution Examples, XIII: Unlucky

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

6. Find $P(X > 2Y \text{ and } Y > X/3)$.

- The region is below.




- Slicing it up gives the leftmost slice at $x = 0$ and the rightmost slice at $x = 3$.
- For each slice, the bottom curve is $y = x/3$ and the upper curve is the line $x = 2y$, which is $y = x/2$.
- Therefore, the integral is

$$\int_0^3 \int_{x/3}^{x/2} \frac{1}{24}(2x + y)\, dy\, dx.$$

# Joint Distribution Examples, XIV: Finally, The End

Example: The continuous random variables $X$ and $Y$ have joint probability density function defined by $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere. Do:

6. Find $P(X > 2Y \text{ and } Y > X/3)$.

- Now we just have to evaluate $\int_0^3 \int_{x/3}^{x/2} \frac{1}{24}(2x + y)\, dy\, dx$.
- This yields

$$\begin{aligned} \int_0^3 \int_0^{x/3} \frac{1}{24}(2x + y)\, dy\, dx &= \int_0^3 \frac{1}{24}(2xy + y^2/2)\Big|_{y=x/3}^{x/2} dx \\ &= \int_0^3 \frac{29}{1728} x^2\, dx \\ &= \frac{29}{5184} x^3 \Big|_{x=0}^{3} = \frac{29}{192}. \end{aligned}$$

# More Variables! (But Let's Not...)

We can also work with joint distributions in more than two variables.

**Definition**

*If $X_1, X_2, \ldots, X_n$ are continuous random variables, then the function $p_{X_1,X_2,\ldots,X_n}(a_1, a_2, \ldots, a_n)$ defined on ordered n-tuples of real numbers, such that $P[(X_1, X_2, \ldots, X_n) \in D] = \int\int \cdots \int_D p_{X_1,X_2,\ldots,X_n}(a_1, a_2, \ldots, a_n)\, da_n da_{n-1} \cdots da_1$ for every region D in n-dimensional space is called the joint probability density function of $X_1, X_2, \ldots, X_n$.*

We will not actually do any of these calculations with more than 2 variables, since the calculations grow substantially in length with 3+ variables.

# Marginal Distributions, I

As in the case of discrete random variables, we can also recover the individual probability distributions $p_X(x)$ and $p_Y(y)$ for either variable from their joint distribution by integrating over the other variable:

**Proposition (Marginal Densities)**

*If $p_{X,Y}(a,b)$ is the joint probability density function for the continuous random variables $X$ and $Y$, then for any $a$ and $b$ we may compute the single-variable probability density functions for $X$ and $Y$ as $p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$ and $p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx$.*

# Marginal Distributions, II

Proof:

- By the definition of the joint probability density function, we know that $P(a \le X \le b) = P(a \le X \le b, -\infty < Y < \infty) = \int_a^b \int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dy\, dx$.
- Thus, we see that integrating $\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dy$ with respect to $x$ on the interval $[a,b]$ yields $P(a \le X \le b)$, which means that $\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dy$ is the probability density function for $X$.
- The second formula follows in the same way upon interchanging the roles of $X$ and $Y$ and switching the order of integration in the iterated integral (this is allowed by a result known as Fubini's theorem, since the integrand is nonnegative).

# Marginal Distributions, III

Example: Let $X$ and $Y$ have joint distribution $p(x,y) = (2x+y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x,y) = 0$ elsewhere.

1. Find the marginal probability density function for $X$.
2. Find $P(1 \le X \le 2)$.
3. Find the marginal probability density function for $Y$.
4. Find $P(0 \le Y \le 1)$.

# Marginal Distributions, III

Example: Let $X$ and $Y$ have joint distribution $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere.

1. Find the marginal probability density function for $X$.
2. Find $P(1 \leq X \leq 2)$.
3. Find the marginal probability density function for $Y$.
4. Find $P(0 \leq Y \leq 1)$.

- To compute the probability density functions, we simply integrate the joint pdf with respect to the other variable.
- Then we can find the requested probabilities using the marginal probability density functions.

# Marginal Distributions, IV

Example: Let $X$ and $Y$ have joint distribution $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere.

1. Find the marginal probability density function for $X$.

# Marginal Distributions, IV

Example: Let $X$ and $Y$ have joint distribution $p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and $p(x, y) = 0$ elsewhere.

1. Find the marginal probability density function for $X$.

   - The marginal pdf for $X$ is given by $p_X(x) = \int_{-\infty}^{\infty} p(x, y)\, dy = \int_0^2 \frac{1}{24}(2x + y)\, dy = \frac{1}{48}(4xy + y^2)\Big|_{y=0}^{2} = \frac{1}{12}(2x + 1)$ for $0 \leq x \leq 3$.

2. Find $P(1 \leq X \leq 2)$.

# Marginal Distributions, IV

Example: Let $X$ and $Y$ have joint distribution $p(x,y) = (2x+y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x,y) = 0$ elsewhere.

1. Find the marginal probability density function for $X$.

- The marginal pdf for $X$ is given by $p_X(x) = \int_{-\infty}^{\infty} p(x,y)\,dy = \int_0^2 \frac{1}{24}(2x+y)\,dy = \frac{1}{48}(4xy+y^2)\Big|_{y=0}^{2} = \frac{1}{12}(2x+1)$ for $0 \le x \le 3$.

2. Find $P(1 \le X \le 2)$.

- This probability is the integral of $p_X(x)$ from $x = 1$ to $x = 2$.
- Explicitly, it is $\int_1^2 \frac{1}{12}(2x+1) = \frac{1}{12}(x^2+x)\big|_{x=1}^{2} = \frac{1}{3}$.

# Marginal Distributions, IV

Example: Let $X$ and $Y$ have joint distribution
$p(x, y) = (2x + y)/24$ for $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and
$p(x, y) = 0$ elsewhere.

3. Find the marginal probability density function for $Y$.

# Marginal Distributions, IV

Example: Let $X$ and $Y$ have joint distribution $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere.

3. Find the marginal probability density function for $Y$.

- The marginal pdf for $Y$ is given by $p_Y(y) = \int_{-\infty}^{\infty} p(x, y)\, dx = \int_0^3 \frac{1}{24}(2x + y)\, dx = \frac{1}{24}(x^2 + xy)\Big|_{x=0}^{3} = \frac{1}{8}(y + 3)$ for $0 \le y \le 2$.

4. Find $P(0 \le Y \le 1)$.

# Marginal Distributions, IV

Example: Let $X$ and $Y$ have joint distribution $p(x, y) = (2x + y)/24$ for $0 \le x \le 3$ and $0 \le y \le 2$, and $p(x, y) = 0$ elsewhere.

3. Find the marginal probability density function for $Y$.
   - The marginal pdf for $Y$ is given by $p_Y(y) = \int_{-\infty}^{\infty} p(x, y)\,dx = \int_0^3 \frac{1}{24}(2x + y)\,dx = \frac{1}{24}(x^2 + xy)\Big|_{x=0}^{3} = \frac{1}{8}(y + 3)$ for $0 \le y \le 2$.
4. Find $P(0 \le Y \le 1)$.
   - This probability is the integral of $p_Y(y)$ from $y = 0$ to $y = 1$.
   - Explicitly, it is $\int_0^1 \frac{1}{8}(y + 3) = \frac{1}{8}(y^2/2 + 3y)|_{y=0}^{1} = \frac{7}{16}$.
   - Note that we actually did this one earlier by setting up the double integral (and of course, we also got $7/16$).

# Summary

We discussed a bit more with Markov's and Chebyshev's inequalities.

We discussed how to set up double integrals and gave several examples.

We introduced joint distributions for continuous random variables and gave several examples of probability calculations.

Next lecture: Independence, covariance, and correlation (again), normal distributions.