Math 3081 (Probability and Statistics)

Lecture #9 of 27 $\sim$ July 20th, 2021

Continuous Random Variables (Part 1.5)

- Computing PDFs and CDFs
- Uniform and Exponential Distributions
- Expected Value
- Variance and Standard Deviation

This material represents §2.2.1-§2.2.3 from the course notes and problems 11-14 from WeBWorK 3.

Recall

Recall our definitions of continuous random variables, PDFs, and CDFs from last time:

Definition

A continuous probability density function is a piecewise-continuous, nonnegative real-valued function $p(x)$ such that $\int_{-\infty}^{\infty} p(x)\,dx = 1$. We say X is a continuous random variable if there exists a continuous probability density function $p(x)$ such that for any interval I on the real line, we have $P(X \in I) = \int_I p(x)\,dx$.

Definition

If X is a continuous random variable with probability density function $p(x)$, its cumulative distribution function (cdf) $c(x)$ is defined as $c(x) = \int_{-\infty}^{x} p(t)\,dt$ for each real value of x.

PDFs + CDFs, I

Example: The probability density function for a continuous random variable X has the form $p(x) = \dfrac{a}{x^{3/2}}$ for $1 \leq x \leq 9$ and 0 elsewhere. Find the following:

1. The value of a.
2. The probability that $1 \leq X \leq 4$.
3. The probability that $4 \leq X \leq 9$.
4. The probability that $X > 0$.
5. The cumulative distribution function for X.
6. The probability that $X \leq 3$.

PDFs + CDFs, II

Example: The probability density function for a continuous random variable X has the form $p(x) = \dfrac{a}{x^{3/2}}$ for $1 \leq x \leq 9$ and 0 elsewhere. Find the following:

1. The value of a.

PDFs + CDFs, II

Example: The probability density function for a continuous random variable X has the form $p(x) = \dfrac{a}{x^{3/2}}$ for $1 \leq x \leq 9$ and 0 elsewhere. Find the following:

1. The value of a.

- The value of a is determined by the fact that the integral of $p(x)$ over its full domain must equal 1.
- In other words, that $\int_1^9 \dfrac{a}{x^{3/2}}\, dx = 1$.
- Since $\int_1^9 \dfrac{a}{x^{3/2}}\, dx = -2ax^{-1/2}\Big|_{x=1}^{9} = \dfrac{4}{3}a$, we see $a = \dfrac{3}{4}$.

PDFs + CDFs, III

Example: The probability density function for a continuous random variable X is $p(x) = \dfrac{3/4}{x^{3/2}}$ for $1 \le x \le 9$. Find:

2. The probability that $1 \le X \le 4$.

PDFs + CDFs, III

<u>Example</u>: The probability density function for a continuous random variable X is $p(x) = \dfrac{3/4}{x^{3/2}}$ for $1 \le x \le 9$. Find:

2. The probability that $1 \le X \le 4$.
   - We simply integrate over the appropriate range.
   - This yields $P(1 \le X \le 4) = \int_1^4 p(x)\,dx = \int_1^4 \dfrac{3}{4} x^{-3/2}\,dx = \dfrac{3}{4}$.
3. The probability that $4 \le X \le 9$.

PDFs + CDFs, III

Example: The probability density function for a continuous random variable X is $p(x) = \frac{3/4}{x^{3/2}}$ for $1 \leq x \leq 9$. Find:

2. The probability that $1 \leq X \leq 4$.
   - We simply integrate over the appropriate range.
   - This yields $P(1 \leq X \leq 4) = \int_1^4 p(x)\,dx = \int_1^4 \frac{3}{4}x^{-3/2}\,dx = \frac{3}{4}$.
3. The probability that $4 \leq X \leq 9$.
   - Integrate: $P(4 \leq X \leq 9) = \int_4^9 p(x)\,dx = \int_4^9 \frac{3}{4}x^{-3/2}\,dx = \frac{1}{4}$.
4. The probability that $X > 0$.

PDFs + CDFs, III

Example: The probability density function for a continuous random variable X is $p(x) = \dfrac{3/4}{x^{3/2}}$ for $1 \le x \le 9$. Find:

2. The probability that $1 \le X \le 4$.
   - We simply integrate over the appropriate range.
   - This yields $P(1 \le X \le 4) = \int_1^4 p(x)\,dx = \int_1^4 \frac{3}{4}x^{-3/2}\,dx = \frac{3}{4}$.
3. The probability that $4 \le X \le 9$.
   - Integrate: $P(4 \le X \le 9) = \int_4^9 p(x)\,dx = \int_4^9 \frac{3}{4}x^{-3/2}\,dx = \frac{1}{4}$.
4. The probability that $X > 0$.
   - We could integrate, but since p is only nonzero for $X \ge 1$, the probability is 1.

PDFs + CDFs, IV

Example: The probability density function for a continuous random variable X is $p(x) = \dfrac{3/4}{x^{3/2}}$ for $1 \leq x \leq 9$. Find:

5. The cumulative distribution function for X.

PDFs + CDFs, IV

Example: The probability density function for a continuous random variable X is $p(x) = \dfrac{3/4}{x^{3/2}}$ for $1 \le x \le 9$. Find:

5. The cumulative distribution function for X.

- We have $c(x) = \int_{-\infty}^{x} p(t)\,dt$. Since the definition of p changes at $x = 1$ and $x = 9$ we will have three cases:

$$c(x) = \begin{cases} 0 & \text{for } x \le 1 \\ \frac{3}{2}(1 - x^{-1/2}) & \text{for } 1 \le x \le 9 \\ 1 & \text{for } x \ge 9 \end{cases}.$$

6. The probability that $X \le 3$.

PDFs + CDFs, IV

Example: The probability density function for a continuous random variable X is $p(x) = \dfrac{3/4}{x^{3/2}}$ for $1 \leq x \leq 9$. Find:

5. The cumulative distribution function for X.

- We have $c(x) = \int_{-\infty}^{x} p(t)\,dt$. Since the definition of p changes at $x = 1$ and $x = 9$ we will have three cases:

$$c(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ \frac{3}{2}(1 - x^{-1/2}) & \text{for } 1 \leq x \leq 9 \\ 1 & \text{for } x \geq 9 \end{cases}.$$

6. The probability that $X \leq 3$.

- We could compute this by integrating, but since we have the cumulative density function, the answer is just $c(3) = \frac{3}{2}(1 - 3^{-1/2})$.

Uniform Distributions, I

A simple class of continuous random variables are those whose probability density functions are constant on an interval $[a, b]$ and zero elsewhere.

- We say such random variables are <u>uniformly distributed</u> on the interval $[a, b]$.
- It is straightforward to see that the pdf and cdf (respectively) for the uniformly-distributed random variable on $[a, b]$ are

$$p(x) = \begin{cases} \dfrac{1}{b-a} & , a \le x \le b \\ 0 & , \text{other } x \end{cases}, \quad c(x) = \begin{cases} 0 & , x < a \\ \dfrac{x-a}{b-a} & , a \le x \le b \\ 1 & , x > b \end{cases}.$$

- Using this description we can easily compute probabilities for uniformly-distributed random variables.

Uniform Distributions, II

Example: The high temperature in Boston in July is uniformly distributed between 70°F and 95°F. Find the probabilities that

1. The temperature is between 82°F and 85°.
2. The temperature is less than 75°F.
3. The temperature is greater than 82°F.
4. The temperature is exactly 77.4821°F.

Uniform Distributions, II

Example: The high temperature in Boston in July is uniformly distributed between 70°F and 95°F. Find the probabilities that

1. The temperature is between 82°F and 85°.
2. The temperature is less than 75°F.
3. The temperature is greater than 82°F.
4. The temperature is exactly 77.4821°F.

- We can evaluate all of these probabilities by integrating the probability density function $p(x) = \begin{cases} 1/25 & \text{for } 70 \leq x \leq 95 \\ 0 & \text{for other } x \end{cases}$.
- More intuitively, we can also find them simply by evaluating the proportion of the total interval that corresponds to the given event.

Uniform Distributions, III

Example: The high temperature in Boston in July is uniformly distributed between 70°F and 95°F. Find the probabilities that

1. The temperature is between 82°F and 85°F.
   - This is $\int_{82}^{85} p(x)\,dx = \int_{82}^{85} \frac{1}{20}\,dx = \frac{3}{25} = 12\%$.
2. The temperature is less than 75°F.
   - This is $\int_{-\infty}^{75} p(x)\,dx = \int_{70}^{75} \frac{1}{25}\,dx = \frac{1}{5} = 20\%$.
3. The temperature is greater than 82°F.
   - This is $\int_{82}^{\infty} p(x)\,dx = \int_{82}^{95} \frac{1}{25}\,dx = \frac{13}{25} = 52\%$.
4. The temperature is exactly 77.4821°F.
   - This is 0, since the probability of any specific single temperature is always 0.

Exponential Distributions, I

Another important class of distributions with a fairly simple definition is the class of exponential distributions:

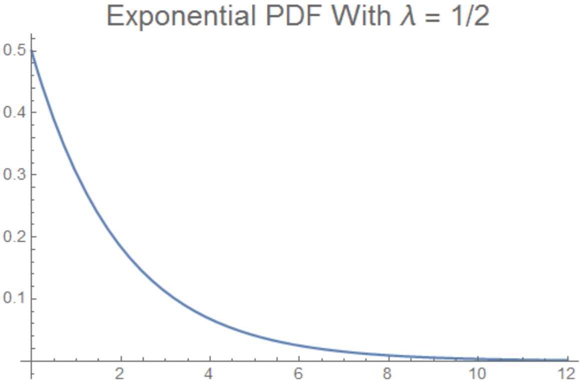
Definition

The exponential distribution with parameter $\lambda > 0$ is the continuous random variable with probability density function $p(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, and is 0 for negative x.

- We can easily compute the cumulative distribution function as
$$c(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x \geq 0 \end{cases}.$$

Exponential Distributions, II

Here is a typical plot of the probability density function of an exponential distribution:



Exponential Distributions, III

Example: If X is exponentially distributed with parameter $\lambda = 1/2$, find the following:

1. $P(X < 1)$.
2. $P(X \geq 3)$.
3. $P(1 \leq X \leq 2)$.
4. $P(X > 10)$.

Exponential Distributions, III

Example: If X is exponentially distributed with parameter $\lambda = 1/2$, find the following:

1. $P(X < 1)$.
2. $P(X \geq 3)$.
3. $P(1 \leq X \leq 2)$.
4. $P(X > 10)$.

- Each of these we can evaluate by integrating the probability density function.
- However, it is easier if we instead use the cumulative distribution function, since we just have to plug the appropriate values into it.

Exponential Distributions, IV

Example: If X is exponentially distributed with parameter $\lambda = 1/2$, find the following:

1. $P(X < 1)$.
2. $P(X \geq 3)$.
3. $P(1 \leq X \leq 2)$.
4. $P(X > 10)$.

- We use $c(x) = 1 - e^{-\lambda x}$ for $x \geq 0$ to get
  1. $P(X < 1) = c(1) = 1 - e^{-1/2} \approx 0.3935$.
  2. $P(X \geq 3) = 1 - c(3) = e^{-3/2} \approx 0.2231$,
  3. $P(1 \leq X \leq 2) = c(2) - c(1) = e^{-1/2} - e^{-1} \approx 0.2387$.
  4. $P(X > 10) = 1 - c(10) = e^{-10/2} \approx 0.0067$.

Expected Value, I

Recall that if X is a discrete random variable, the expected value $E(X)$ is defined as $E(X) = \sum_{s_i \in S} p_X(s_i) X(s_i)$.

- In words, we compute the expected value by summing the possible values of X over all the outcomes in the sample space, weighted by their probabilities as measured by the probability distribution function $p_X(x)$.
- Equivalently, when the sample space is a set of real numbers $S = \{x_1, x_2, \ldots, x_n\}$, this sum can be equivalently written as $E(X) = \sum_{i=1}^{n} p_X(x_i) \cdot x_i$.
- The natural analogue for continuous random variables is to convert the sum into an integral, keep the probability distribution function for X as is, and think of x_i as the variable x. (If you like, you could also recognize the expression above as a Riemann sum for an integral.)

Expected Value, II

This leads us to the following definition:

Definition

If X is a continuous random variable with probability density function $p(x)$, we define the expected value as $E(X) = \int_{-\infty}^{\infty} x\, p(x)\, dx$, presuming that the integral converges.

We interpret the expected value in the same way as before: if we sample the random variable a large number of times, the average value of the sample will approach the expected value as the sample size goes to infinity.

Expected Value, III

Examples: Find the expected values of the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.
2. Q, with probability density function $q(x) = 2x$ for $0 \le x \le 1$.
3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \ge 0$.
4. R, with probability density function $r(x) = 1/x^2$ for $x \ge 1$.
5. C, with probability density function $c(x) = \dfrac{1}{\pi(1 + x^2)}$.

Expected Value, III

Examples: Find the expected values of the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.
2. Q, with probability density function $q(x) = 2x$ for $0 \le x \le 1$.
3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \ge 0$.
4. R, with probability density function $r(x) = 1/x^2$ for $x \ge 1$.
5. C, with probability density function $c(x) = \dfrac{1}{\pi(1 + x^2)}$.

- We simply need to compute the required integral in each case.

Expected Value, IV

Examples: Find the expected values of the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \leq x \leq 1$.

Expected Value, IV

Examples: Find the expected values of the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.
   - By definition, we have $E(P) = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_0^1 x \cdot 6(x - x^2)\, dx = (2x^3 - \frac{3}{2}x^4)\big|_{x=0}^{1} = \dfrac{1}{2}$.
2. Q, with probability density function $q(x) = 2x$ for $0 \le x \le 1$.

Expected Value, IV

Examples: Find the expected values of the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.
   - By definition, we have $E(P) = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_0^1 x \cdot 6(x - x^2)\, dx = (2x^3 - \frac{3}{2}x^4)\big|_{x=0}^{1} = \dfrac{1}{2}$.
2. Q, with probability density function $q(x) = 2x$ for $0 \le x \le 1$.
   - Similarly, $E(Q) = \int_{-\infty}^{\infty} x\, q(x)\, dx = \int_0^1 x \cdot 2x\, dx = (\frac{2}{3}x^3)\big|_{x=0}^{1} = \dfrac{2}{3}$.

Expected Value, V

Examples: Find the expected values of the following continuous random variables:

3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.

Expected Value, V

<u>Examples</u>: Find the expected values of the following continuous random variables:

3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.
   - We compute $E(E_\lambda) = \int_{-\infty}^{\infty} x\, e(x)\, dx = \int_0^\infty \lambda x\, e^{-\lambda x}\, dx = \left(-xe^{-\lambda x} - e^{-\lambda x}/\lambda\right)\Big|_{x=0}^{\infty} = 1/\lambda$.
   - Notice that the integral was improper. (But that's fine!)
   - This tells us that the exponential distribution with parameter λ has expected value $1/\lambda$.
4. R, with probability density function $r(x) = 1/x^2$ for $x \geq 1$.

Expected Value, V

Examples: Find the expected values of the following continuous random variables:

3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.
   - We compute $E(E_\lambda) = \int_{-\infty}^{\infty} x\, e(x)\, dx = \int_0^{\infty} \lambda x\, e^{-\lambda x}\, dx = (-xe^{-\lambda x} - e^{-\lambda x}/\lambda)\big|_{x=0}^{\infty} = 1/\lambda$.
   - Notice that the integral was improper. (But that's fine!)
   - This tells us that the exponential distribution with parameter λ has expected value $1/\lambda$.
4. R, with probability density function $r(x) = 1/x^2$ for $x \geq 1$.
   - We compute
     $$E(R) = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_1^{\infty} x \cdot \frac{1}{x^2}\, dx = \ln(x)\big|_{x=1}^{\infty} = \infty.$$
   - Notice that this integral was also improper, and ended up having value ∞. This random variable shows that, just like in the discrete case, we can have an infinite expected value.

Expected Value, VI

<u>Examples</u>: Find the expected values of the following continuous random variables:

5. C, with probability density function $c(x) = \dfrac{1}{\pi(1+x^2)}$.

Expected Value, VI

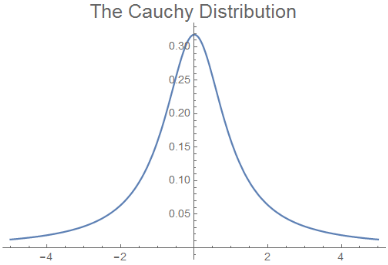
Examples: Find the expected values of the following continuous random variables:

5. C, with probability density function $c(x) = \dfrac{1}{\pi(1+x^2)}$.

- We compute $E(C) = \int_{-\infty}^{\infty} x \cdot \dfrac{1}{\pi(1+x^2)}\,dx$.
- Like the last two examples, this integral is also improper.
- To evaluate it, we split the range of integration at 0 to obtain $\int_0^{\infty} x \cdot \dfrac{1}{\pi(1+x^2)}\,dx = \frac{1}{2\pi}\ln(1+x^2)\Big|_{x=0}^{\infty} = \infty$, while $\int_{-\infty}^{0} x \cdot \dfrac{1}{\pi(1+x^2)}\,dx = \frac{1}{2\pi}\ln(1+x^2)\Big|_{x=-\infty}^{0} = -\infty$.
- But this tells us that the original integral $E(C) = \int_{-\infty}^{\infty} x \cdot \dfrac{1}{\pi(1+x^2)}\,dx$ is $\infty - \infty$, which is not defined.
- This means that the expected value of C does not exist.

Expected Value, VII

The probability distribution in the last example is called the Cauchy distribution. Here is a plot of its pdf:



Observe the strange fact that the expected value does not exist, despite the fact that the distribution is symmetric around $x = 0$ (and thus, one would think, the average value should be 0!).

The Cauchy distribution yields counterexamples to several "obvious-seeming" false statements like that one.

Expected Value of Functions of X, I

Motivated by our successful definition of the expected value, we would now like to define the variance and standard deviation.

- We can attempt to pose the definition in exactly the same way as for discrete random variables: namely, as $\mathrm{var}(X) = E[(X - \mu)^2]$ where $\mu = E(X)$ is the expected value of X, or equivalently as $\mathrm{var}(X) = E(X^2) - [E(X)]^2$ using the "alternate" formula for the variance.
- However, we need to know how to compute the expected value of a function of X.
- It is not so obvious how to do this directly, even in very simple examples, as we will now illustrate.

Expected Value of Functions of X, II

Suppose X is uniformly distributed on $[0, 2]$: we want to compute the variance of X, which requires finding $E(X^2)$.

- Since X^2 is a random variable, it has some probability density function, which we can try to calculate by using the cumulative distribution function.
- Explicitly, since $X^2 \le a$ is equivalent to $X \le \sqrt{a}$ (at least for X nonnegative), this means that $c_{X^2}(a) = c_X(\sqrt{a})$ for $0 \le a \le 4$.
- In terms of the probability density functions, this says $\int_0^a p_{X^2}(x)\,dx = \int_0^{\sqrt{a}} p_X(x)\,dx = \int_0^{\sqrt{a}} (1/2)\,dx = \sqrt{a}/2$.
- Then differentiating both sides yields $p_{X^2}(a) = \dfrac{d}{da}\left[\sqrt{a}/2\right] = \frac{1}{4}a^{-1/2}$ for $0 \le a \le 4$.
- This gives us everything we need to write down the probability density function for p_{X^2}.

Expected Value of Functions of X, III

- To summarize the previous slide, we have
$$p_{X^2}(x) = \begin{cases} x^{-1/2}/4 & \text{for } 0 \le x \le 4 \\ 0 & \text{for other } x \end{cases}.$$
- We can then evaluate
$$E(X^2) = \int_0^4 x \cdot \frac{1}{4} x^{-1/2}\, dx = \frac{1}{6} x^{3/2} \Big|_{x=0}^{4} = \frac{4}{3}.$$
- Finally, this allows us to compute the variance of the original random variable X, namely, as
$$\mathrm{var} X = E(X^2) - E(X)^2 = \frac{4}{3} - 1^2 = \frac{1}{3}.$$

This was quite difficult, even for the simplest possible distribution! We would like a better approach.

Expected Value of Functions of X, IV

Fortunately, we can give a better approach!

- Consider instead the case of a discrete random variable X taking values $x_1, x_2, \ldots$ with probabilities $p_1, p_2, \ldots$.
- Then if g is any function, $g(X)$ takes values $g(x_1), g(x_2), \ldots$ with probabilities $p_1, p_2, \ldots$, and so $E[g(X)] = g(x_1)p_1 + g(x_2)p_2 + \cdots = \sum_i g(x_i)p_i$.
- Now we can simply write down the continuous analogue of this formula:

Expected Value of Functions of X, IV

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- Consider instead the case of a discrete random variable X taking values $x_1, x_2, \ldots$ with probabilities $p_1, p_2, \ldots$.
- Then if g is any function, $g(X)$ takes values $g(x_1), g(x_2), \ldots$ with probabilities $p_1, p_2, \ldots$, and so $E[g(X)] = g(x_1)p_1 + g(x_2)p_2 + \cdots = \sum_i g(x_i)p_i$.
- Now we can simply write down the continuous analogue of this formula:

Proposition (Expected Value of Functions of X)

If X is a continuous random variable with probability density function $p(x)$, and $g(x)$ is any piecewise-continuous function, then the expected value of $g(X)$ is $E[g(X)] = \int_{-\infty}^{\infty} g(x)\, p(x)\, dx$.

Expected Value of Functions of X, V

<u>Proof</u> (special case):

- Suppose g is increasing and has an inverse function g^{-1}.
- Then $g(x) \leq a$ is equivalent to $x \leq g^{-1}(a)$, so by the same argument we gave earlier, $c_{g(X)}(a) = c_X(g^{-1}(a))$.
- Differentiating both sides yields $p_{g(X)}(a) = p(g^{-1}(a)) \cdot \dfrac{1}{g'(g^{-1}(a))}$, and then $E[g(X)] = \int_{-\infty}^{\infty} x \cdot (g^{-1}(x)) \cdot \dfrac{1}{g'(g^{-1}(x))}\, dx$.
- Making the substitution $u = g^{-1}(x)$, so that $x = g(u)$ and $dx = g'(u)du$, in the integral and simplifying yields $E[g(X)] = \int_{-\infty}^{\infty} g(u) \cdot p(u)\, du$, as claimed.

Expected Value of Functions of X, VI

Example: If X is uniformly distributed on $[0, 2]$, find the following:

1. $E(X)$.
2. $E(X^2)$.
3. $E(\sqrt{4X+1})$.
4. $E(e^X)$.

Expected Value of Functions of X, VI

Example: If X is uniformly distributed on $[0, 2]$, find the following:

1. $E(X)$.
2. $E(X^2)$.
3. $E(\sqrt{4X+1})$.
4. $E(e^X)$.

- Using the probability density function $p(x) = 1/2$ for $0 \le x \le 2$, we simply have to evaluate the appropriate integrals for each of these.
- Specifically: $E(g(X)) = \int_0^2 g(x) \cdot (1/2)\, dx$.

Expected Value of Functions of X, VII

Example: If X is uniformly distributed on $[0, 2]$, find the following:

1. $E(X)$.
   - We have $E(X) = \int_0^2 x \cdot (1/2)\, dx = \frac{1}{4}x^2\Big|_{x=0}^{2} = 1$.
2. $E(X^2)$.
   - We have $E(X^2) = \int_0^2 x^2 \cdot (1/2)\, dx = \frac{1}{6}x^3\Big|_{x=0}^{2} = 4/3$.
3. $E(\sqrt{4X+1})$.
   - We have $E(\sqrt{4X+1}) = \int_0^2 \sqrt{4x+1} \cdot (1/2)\, dx = \frac{1}{12}(4x+1)^{3/2}\Big|_{x=0}^{2} = 13/6$.
4. $E(e^X)$.
   - We have $E(e^X) = \int_0^2 e^x \cdot (1/2)\, dx = e^x/2\Big|_{x=0}^{2} = (e^2 - 1)/2$.

Properties of Expected Value: The Sequel

Expected value has the same properties as it did before:

Corollary (Linearity of Expected Value)

If X and Y are continuous random variables whose expected values are defined, and a and b are any real numbers, then $E(aX + b) = a \cdot E(X) + b$ and $E(X + Y) = E(X) + E(Y)$.

Properties of Expected Value: The Sequel

Expected value has the same properties as it did before:

Corollary (Linearity of Expected Value)

If X and Y are continuous random variables whose expected values are defined, and a and b are any real numbers, then $E(aX + b) = a \cdot E(X) + b$ *and* $E(X + Y) = E(X) + E(Y)$.

<u>Proof</u> (of first statement):

- If X has probability density function $p(x)$, then $E(aX + b) = \int_{-\infty}^{\infty}(ax + b) \cdot p(x)\, dx =$ $a \int_{-\infty}^{\infty} x \cdot p(x) + b \int_{-\infty}^{\infty} p(x)\, dx = a \cdot E(X) + b.$

The second statement can be deduced using a similar approach. However, it requires using joint distributions, so we won't give the argument right now.

Variance and Standard Deviation, I

Now we can properly define the variance and standard deviation:

Definition

If X is a continuous random variable whose expected value μ exists and is finite, the <u>*variance*</u> $\mathrm{var}(X)$ *is defined as* $\mathrm{var}(X) = E[(X-\mu)^2] = E(X^2) - E(X)^2$.

- The equality $E[(X-\mu)^2] = E(X^2) - E(X)^2$ follows for continuous random variables by the same argument used for discrete random variables.

Definition

If X is a continuous random variable whose variance exists, its <u>*standard deviation*</u> *is* $\sigma(X) = \sqrt{\mathrm{var}(X)}$.

Variance and Standard Deviation, II

Example: Find the variance and standard deviation for the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.
2. Q, with probability density function $q(x) = 2x$ for $0 \le x \le 1$.
3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \ge 0$.
4. S, with probability density function $s(x) = 2/x^3$ for $x \ge 1$.

Variance and Standard Deviation, II

Example: Find the variance and standard deviation for the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.
2. Q, with probability density function $q(x) = 2x$ for $0 \le x \le 1$.
3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \ge 0$.
4. S, with probability density function $s(x) = 2/x^3$ for $x \ge 1$.

- For each of these, we use the same approach as for discrete random variables: we compute $E(X)$ and $E(X^2)$, then plug in to $\mathrm{var}(X) = E(X^2) - E(X)^2$.

Variance and Standard Deviation, III

Example: Find the variance and standard deviation for the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.

Variance and Standard Deviation, III

Example: Find the variance and standard deviation for the following continuous random variables:

1. P, with probability density function $p(x) = 6(x - x^2)$ for $0 \le x \le 1$.

- We have $E(P) = \int_0^1 x \cdot 6(x - x^2)\, dx = (2x^3 - \frac{3}{2}x^4)\big|_{x=0}^1 = 1/2$.
- Also, $E(P^2) = \int_0^1 x^2 \cdot 6(x - x^2)\, dx = (\frac{3}{2}x^4 - \frac{6}{5}x^5)\big|_{x=0}^1 = 3/10$.
- Thus $\mathrm{var}(P) = E(P^2) - [E(P)]^2 = (3/10) - (1/2)^2 = 1/20$ and $\sigma(P) = \sqrt{1/20}$.

Variance and Standard Deviation, IV

Example: Find the variance and standard deviation for the following continuous random variables:

2. Q, with probability density function $q(x) = 2x$ for $0 \leq x \leq 1$.

Variance and Standard Deviation, IV

Example: Find the variance and standard deviation for the following continuous random variables:

2. Q, with probability density function $q(x) = 2x$ for $0 \le x \le 1$.

- We have $E(Q) = \int_0^1 x \cdot 2x\,dx = \left(\frac{2}{3}x^3\right)\Big|_{x=0}^{1} = 2/3$.
- Also, $E(Q^2) = \int_0^1 x^2 \cdot 2x\,dx = \left(\frac{1}{2}x^4\right)\Big|_{x=0}^{1} = 1/2$.
- Thus $\operatorname{var}(Q) = E(Q^2) - [E(Q)]^2 = 1/2 - (2/3)^2 = 1/18$ and $\sigma(Q) = \sqrt{1/18}$.

Variance and Standard Deviation, V

Example: Find the variance and standard deviation for the following continuous random variables:

3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.

Variance and Standard Deviation, V

Example: Find the variance and standard deviation for the following continuous random variables:

3. E_λ, with probability density function $e_\lambda(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.

- We have $E(E_\lambda) = \int_0^\infty \lambda x\, e^{-\lambda x}\, dx = 1/\lambda$.
- Also, $E(E_\lambda^2) = \int_0^\infty x^2 \lambda\, e^{-\lambda x}\, dx = -(x^2 + 2x/\lambda + 2/\lambda^2)e^{-\lambda x}\Big|_{x=0}^{\infty} = 2/\lambda^2$.
- Thus $\mathrm{var}(E_\lambda) = E(E_\lambda^2) - [E(E_\lambda)]^2 = 2/\lambda^2 - (1/\lambda)^2 = 1/\lambda^2$ and $\sigma(E_\lambda) = 1/\lambda$.
- This tells us that the expected value and standard deviation of the exponential distribution with parameter λ are both $1/\lambda$.

Variance and Standard Deviation, VI

Example: Find the variance and standard deviation for the following continuous random variables:

4. S, with probability density function $s(x) = 2/x^3$ for $x \geq 1$.

Variance and Standard Deviation, VI

Example: Find the variance and standard deviation for the following continuous random variables:

4. S, with probability density function $s(x) = 2/x^3$ for $x \geq 1$.

- We have $E(S) = \int_1^\infty x \cdot (2/x^3)\,dx = (-\frac{2}{x})\big|_{x=1}^\infty = 2$.
- Also, $E(S^2) = \int_1^\infty x^2 \cdot (2/x^3)\,dx = (2\ln x)\big|_{x=1}^\infty = \infty$.
- Thus $\mathrm{var}(S) = E(S^2) - [E(S)]^2 = \infty - 2^2 = \infty$ and $\sigma(S) = \infty$ as well.

Here, we have an example of a continuous random variable with finite expected value but infinite variance.

Properties of Variance

Just like with properties of expected value, we get the same properties of variance in the continuous case:

Proposition (Properties of Variance)

If X is a continuous random variable and a and b are any real numbers, then $\mathrm{var}(aX + b) = a^2 \cdot \mathrm{var}(X)$ and $\sigma(aX + b) = |a|\,\sigma(X)$.

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Proof: Identical to the case for discrete random variables.

Example: If X is a continuous random variable with expected value 4 and standard deviation 3, what are the expected value and standard deviation of $3X - 5$?

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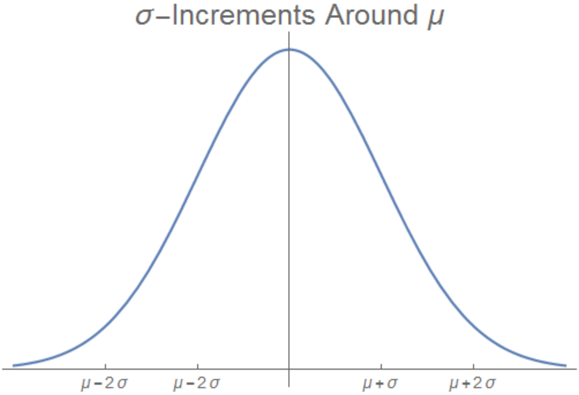
<u>Proof</u>: Identical to the case for discrete random variables.

<u>Example</u>: If X is a continuous random variable with expected value 4 and standard deviation 3, what are the expected value and standard deviation of $3X - 5$?

- From the properties of expected value and variance, we have $E(3X - 5) = 3E(X) - 5 = 7$ and $\sigma(3X - 5) = |3|\,\sigma(X) = 9$.

Chebyshev and Markov's Inequalities, I

Intuitively, we should expect that "most" of the distribution for a random variable X should be concentrated near its average, provided that we measure in increments of the standard deviation:



Chebyshev and Markov's Inequalities, II

We can make this statement more precise, as follows:

Theorem (Chebyshev's Inequality)

If X is a random variable with expected value μ and standard deviation σ, then $P(|X - \mu| \geq k\sigma) \leq 1/k^2$ for any positive real number k.

- In words, Chebyshev's inequality says that the probability that X takes a value at least k standard deviations away from its mean is at most $1/k^2$.
- For $k = 2$, it says that the value of X is 2 or more standard deviations away from the mean at most $1/4$ of the time.
- Similarly, for $k = 3$ it says that the value is 3 or more standard deviations away from the mean at most $1/9$ of the time.

Chebyshev and Markov's Inequalities, III

To prove Chebyshev's inequality, we first prove a lemma called Markov's inequality:

Lemma (Markov's Inequality)

If Y is a nonnegative random variable and a is any positive real number, then $P(Y \geq a) \leq E(Y)/a$.

Chebyshev and Markov's Inequalities, III

To prove Chebyshev's inequality, we first prove a lemma called Markov's inequality:

Lemma (Markov's Inequality)

If Y is a nonnegative random variable and a is any positive real number, then $P(Y \geq a) \leq E(Y)/a$.

Proof (of Markov's inequality):

- We break the expected value calculation into the two pieces where $0 \leq Y < a$ and $Y \geq a$.
- Notice that the expected value of Y when $0 \leq Y < a$ is at least 0, and the expected value of Y when $Y \geq a$ is at least a.
- Therefore, $E(Y) = P(0 \leq Y < a) \cdot E(Y|0 \leq Y < a)$ $+ P(Y \geq a) \cdot E(Y|Y \geq a) \geq P(Y < a) \cdot 0 + P(Y \geq a) \cdot a$.
- This means $E(Y) \geq P(Y \geq a) \cdot a$, so $P(Y \geq a) \leq E(Y)/a$.

Chebyshev and Markov's Inequalities, IV

Proof (of Chebyshev's inequality):

Chebyshev and Markov's Inequalities, IV

Proof (of Chebyshev's inequality):

- Apply Markov's inequality to the random variable $Y = (X - \mu)^2$ and $a = k^2\sigma^2$ (note that $Y \geq 0$ and $a > 0$ here so the result applies).
- The inequality says $P[(X - \mu)^2 \geq k^2\sigma^2) \leq E[(X - \mu)^2]/(k^2\sigma^2)$.
- But since $E[(X - \mu)^2] = \sigma^2$, we obtain $P[(X - \mu)^2 \geq k^2\sigma^2) \leq \sigma^2/(k^2\sigma^2) = 1/k^2$.
- Since $(X - \mu)^2 \geq k^2\sigma^2$ is equivalent to $|X - \mu| \geq k\sigma$, we have obtained Chebyshev's inequality.

Notice that we only used properties of expected value in this proof, so in fact Chebyshev's inequality holds for any random variable, discrete or continuous!

Chebyshev and Markov's Inequalities, V

Chebyshev's inequality gives a precise bound on how far away from its mean, in terms of its standard deviation, the probability distribution of a random variable can be concentrated.

- For almost all distributions, Chebyshev's inequality is very conservative (relative to reality): most distributions actually lie within 2 standard deviations of the mean much more than 75% of the time.
- However, for the discrete random variable taking the values -1, 0, and 1 with respective probabilities $1/(2t^2)$, $1 - 1/t^2$, and $1/(2t^2)$, the mean is 0 and the standard deviation is $1/t$, so the inequality is sharp for this distribution and $k = t$.

Chebyshev and Markov's Inequalities, VI

<u>Example</u>: For the uniform distribution on $[0, 2]$, determine what proportion of the distribution lies between $k = 1, 1.5, 2, 2.5$ standard deviations of the mean and compare the results to the bounds from Chebyshev's inequality.

- We have previously computed $\mu = 1$ and $\sigma = \sqrt{1/3} \approx 0.5774$ for this distribution.
- So we are seeking $P(|X - \mu| < k\sigma)$ for these values of k.
- Here are the results, along with the lower bounds from Chebyshev's inequality:

k	1	1.5	2	2.5
$P(\|X - \mu\| < k\sigma)$	0.577	0.866	1	1
Chebyshev Bound $(1 - 1/k^2)$	0	0.555	0.750	0.840

Summary

We introduced continuous random variables and their associated probability density functions and cumulative distribution functions.

We defined the expected value of a continuous random variable and established some of its properties.

We discussed the variance and standard deviation of a continuous random variable.

Next lecture: More with Markov and Chebyshev's inequalities, joint distributions for continuous random variables.