Math 3081 (Probability and Statistics)

Lecture #4 of 27 $\sim$ July 12th, 2021

Conditional Probability + Independence

- Conditional Probability
- Computing With Conditional Probability
- Independence

This material represents §1.4.1-§1.4.2 from the course notes and problems 1-8 from WeBWorK 2.

Recall

Last week, we introduced the notion of a probability distribution and discussed how to compute probabilities under the assumption of "fairness", in which all outcomes in the sample space are equally likely. We also introduced conditional probability:

Definition

If A and B are events and $P(B) > 0$, we define the conditional probability $P(A|B)$, the probability that A occurs given that B occurred, as $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Our goal now is to work out some more examples of conditional probabilities, and then explain how to use them to solve more general probability problems.

More Conditional Probability, I

Example: Suppose four fair coins are flipped. Determine the probabilities of the respective events

1. There are exactly two heads, given the first flip is a tail.
2. The first flip is heads, given there are exactly 3 heads.
3. All four flips are tails, given there is at least one tail.
4. There are at least 3 heads, given there are at least 2 heads.

More Conditional Probability, I

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4. There are at least 3 heads, given there are at least 2 heads.

- Each of these is an example of a conditional probability.
- To compute them, we use the formula $P(A|B) = P(A \cap B) \,/\, P(B)$.
- This requires us to identify the events A and B and then to compute $P(B)$ and $P(A \cap B)$.

More Conditional Probability, II

<u>Example</u>: Suppose four fair coins are flipped. Determine the probabilities of the respective events

1. There are exactly two heads, given the first flip is a tail.

More Conditional Probability, II

Example: Suppose four fair coins are flipped. Determine the probabilities of the respective events

1. There are exactly two heads, given the first flip is a tail.

- This probability is of the form $P(A|B)$ where A is the event that there are exactly two heads and B is the event that the first flip is a tail.
- We can see that $P(B) = 1/2$ because 8 of the 16 possible outcomes have the first flip a tail.
- Also, $A \cap B$ is the event that the first flip is a tail and there are exactly two heads, which occurs in three ways: $\{THHT, THTH, TTHH\}$. Thus, $P(A \cap B) = 3/16$.
- Finally, by putting all of this together, we see $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{3/16}{1/2} = \dfrac{3}{8}$.

More Conditional Probability, III

Example: Suppose four fair coins are flipped. Determine the probabilities of the respective events

2. The first flip is heads, given there are exactly 3 heads.

More Conditional Probability, III

Example: Suppose four fair coins are flipped. Determine the probabilities of the respective events

2. The first flip is heads, given there are exactly 3 heads.

- This probability is of the form $P(A|B)$ where A is the event that the first flip is heads and B is the event that there are exactly three heads.
- We can see that $P(B) = 4/16 = 1/4$ because there are $\binom{4}{3} = 4$ ways to flip three heads.
- Also, $A \cap B$ is the event that the first flip is heads and there are exactly three heads, which occurs in three ways: $\{HHHT, HHTH, HTHH\}$. Thus, $P(A \cap B) = 3/16$.
- Finally, by putting all of this together, we see $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{3/16}{1/4} = \dfrac{3}{4}$.

More Conditional Probability, IV

Example: Suppose four fair coins are flipped. Determine the probabilities of the respective events

3. All four flips are tails, given there is at least one tail.

More Conditional Probability, IV

<u>Example</u>: Suppose four fair coins are flipped. Determine the probabilities of the respective events

3. All four flips are tails, given there is at least one tail.

- This probability is of the form $P(A|B)$ where A is the event that all four flips are tails and B is the event that there is at least one tail.
- We can see that $P(B) = 15/16$ because B^c is the event that there are no tails, which only occurs in one way (hence has probability $1/16$).
- Also, $A \cap B$ is simply the event A, which has probability $1/16$. Thus, $P(A \cap B) = 1/16$.
- Finally, by putting all of this together, we see $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/16}{15/16} = \dfrac{1}{15}$.

More Conditional Probability, V

<u>Example</u>: Suppose four fair coins are flipped. Determine the probabilities of the respective events

4. There are at least 3 heads, given there are at least 2 heads.

More Conditional Probability, V

<u>Example</u>: Suppose four fair coins are flipped. Determine the probabilities of the respective events

4. There are at least 3 heads, given there are at least 2 heads.

- This probability is of the form $P(A|B)$ where A is the event that there are at least three heads and B is the event that there are at least two heads.
- The event B occurs when there are 2, 3, or 4 heads, which have $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$ possibilities, respectively, for a total of 11. Thus, $P(B) = 11/16$.
- $A \cap B$ occurs when there are 3 or 4 heads, which have $\binom{4}{3}$ and $\binom{4}{4}$ possibilities for a total of 5. Thus $P(A \cap B) = 5/16$.
- Therefore $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{5/16}{11/16} = \dfrac{5}{11}$.

More Conditional Probability, VI

Example: Suppose four fair coins are flipped. Determine the probabilities of the respective events

1. There are exactly two heads, given the first flip is a tail: $3/8$.
2. The first flip is heads, given there are exactly 3 heads: $3/4$.
3. All four flips are tails, given there is at least one tail: $1/15$.
4. There are at least 3 heads, given there are at least 2 heads: $5/11$.

More Conditional Probability, VI

<u>Example</u>: Suppose four fair coins are flipped. Determine the probabilities of the respective events

1. There are exactly two heads, given the first flip is a tail: $3/8$.
2. The first flip is heads, given there are exactly 3 heads: $3/4$.
3. All four flips are tails, given there is at least one tail: $1/15$.
4. There are at least 3 heads, given there are at least 2 heads: $5/11$.

It is also possible to solve each of these problems by enumerating the possible outcomes explicitly.

- For (1): there are 8 ways in which the first flip can be a tail, and in 3 of these, there are exactly 2 heads.
- For (2): there are 4 ways in which there are exactly three heads, and in 3 of these, the first flip is heads.
- The others are similar. (Try them yourself!)

More Conditional Probability, VII

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

1. $P(B^c)$.
2. $P(A \cap B)$.
3. $P(A \cup B)$.
4. $P(B|A)$.
5. $P(A \cap B^c)$.
6. $P(A|B^c)$.
7. $P(A^c \cap B^c)$.

More Conditional Probability, VII

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

1. $P(B^c)$.
2. $P(A \cap B)$.
3. $P(A \cup B)$.
4. $P(B|A)$.
5. $P(A \cap B^c)$.
6. $P(A|B^c)$.
7. $P(A^c \cap B^c)$.

It can be very helpful to use a Venn diagram in solving problems that involve unions or intersections of events, like this problem does.

More Conditional Probability, VIII

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

1. $P(B^c)$.

More Conditional Probability, VIII

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

1. $P(B^c)$.
   - Using the complement formula, we see that $P(B^c) = 1 - P(B) = 1 - 0.2 = 0.8$.
2. $P(A \cap B)$.

More Conditional Probability, VIII

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

1. $P(B^c)$.
   - Using the complement formula, we see that $P(B^c) = 1 - P(B) = 1 - 0.2 = 0.8$.
2. $P(A \cap B)$.
   - The conditional probability formula says that $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.
   - Rearranging the formula yields $P(A \cap B) = P(A|B) \cdot P(B) = 0.8 \cdot 0.2 = 0.16$.

More Conditional Probability, IX

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

3. $P(A \cup B)$.

More Conditional Probability, IX

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

3. $P(A \cup B)$.
   - By the union-intersection probability formula, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.2 - 0.16 = 0.44$.
4. $P(B|A)$.

More Conditional Probability, IX

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

3. $P(A \cup B)$.
   - By the union-intersection probability formula, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.2 - 0.16 = 0.44$.
4. $P(B|A)$.
   - Using the conditional probability formula, with A and B interchanged, we see that $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.16}{0.4} = 0.4$.

More Conditional Probability, X (Marks The Spot)

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

5. $P(A \cap B^c)$.

More Conditional Probability, X (Marks The Spot)

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

5. $P(A \cap B^c)$.

- The idea is to notice that A is the union of the two disjoint events $A \cap B$ and $A \cap B^c$. (These two events are the two pieces in the Venn diagram that make up the region for A.)
- Therefore, $P(A \cap B) + P(A \cap B^c) = P(A)$, so $P(A \cap B^c) = P(A) - P(A \cap B) = 0.4 - 0.16 = 0.24$.

6. $P(A|B^c)$.

More Conditional Probability, X (Marks The Spot)

Example: Suppose A and B are events such that $P(A) = 0.4$, $P(A|B) = 0.8$, and $P(B) = 0.2$. Find the following:

5. $P(A \cap B^c)$.

- The idea is to notice that A is the union of the two disjoint events $A \cap B$ and $A \cap B^c$. (These two events are the two pieces in the Venn diagram that make up the region for A.)
- Therefore, $P(A \cap B) + P(A \cap B^c) = P(A)$, so $P(A \cap B^c) = P(A) - P(A \cap B) = 0.4 - 0.16 = 0.24$.

6. $P(A|B^c)$.

- Using the conditional probability formula with A and B^c, we see that $P(A|B^c) = \dfrac{P(A \cap B^c)}{P(B^c)} = \dfrac{0.24}{0.8} = 0.3$.

Independence, I

We now develop the notion of when two events A and B are independent of one another.

- Intuitively, we would say that A and B are independent if knowing information about one of them does not provide any information about the other one.
- This is something we can easily rephrase in terms of conditional probabilities:

Definition

We say that two events A and B are <u>independent</u> if $P(A|B) = P(A)$, or equivalently, if $P(B|A) = P(B)$. Two events that are not independent are said to be <u>dependent</u>.

Independence, II

We can rearrange the definition of independence in a different way using the definition of the conditional probability $P(A|B)$:

- Specifically, since $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, saying that $P(A|B) = P(A)$ is the same as saying that $P(A \cap B) = P(A) \cdot P(B)$ after clearing the denominator.
- This observation also allows us to see that the two statements of independence are equivalent to each other, since both $P(A|B) = P(A)$ and $P(B|A) = P(B)$ are equivalent to $P(A \cap B) = P(A) \cdot P(B)$.

To summarize, A and B are independent precisely when $P(A \cap B) = P(A) \cdot P(B)$.

Independence, III

Example: Suppose two fair 6-sided dice are rolled. Determine whether the events A that the first roll is a 2 and B that the second roll is a 5 are independent.

Independence, III

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- We simply need to decide if $P(A \cap B) = P(A) \cdot P(B)$.
- There are 36 equally-likely outcomes from rolling the two dice. In 6 of these A occurs, in 6 of these B occurs, and in 1 of these $A \cap B$ occurs.
- Thus, $P(A) = 1/6$, $P(B) = 1/6$, and $P(A \cap B) = 1/36$.
- Since indeed $P(A \cap B) = 1/36 = P(A) \cdot P(B)$, the events A and B are independent.

This finding of independence is very reasonable: event A only concerns the first roll while event B only concerns the second roll, and these two rolls do not affect one another.

Independence, IV

Example: Suppose two fair 6-sided dice are rolled. Determine whether the events A that the first roll is a 2 and C that the sum of the two rolls is 3 are independent.

Independence, IV

Example: Suppose two fair 6-sided dice are rolled. Determine whether the events A that the first roll is a 2 and C that the sum of the two rolls is 3 are independent.

- We simply need to decide if $P(A \cap C) = P(A) \cdot P(C)$.
- There are 36 equally-likely outcomes from rolling the two dice. In 6 of these A occurs, in 2 of these C occurs (1-2 or 2-1), and in 1 of these $A \cap C$ occurs (2-1).
- Thus, $P(A) = 1/6$, $P(C) = 1/18$, and $P(A \cap C) = 1/36$.
- We see $P(A \cap C) = 1/36$ while $P(A) \cdot P(C) = 1/108$. Since these are not equal, the events A and C are not independent.

This finding of non-independence is very reasonable: knowing that the first roll was a 2 (instead of, say, a 6) makes it possible for C to occur, and (conversely), knowing that the sum was 3 makes it much more likely that the first roll was a 2.

Independence, V

Example: Suppose two fair 6-sided dice are rolled. Determine whether the events B that the second roll is a 5 and C that the sum of the two rolls is 3 are independent.

Independence, V

Example: Suppose two fair 6-sided dice are rolled. Determine whether the events B that the second roll is a 5 and C that the sum of the two rolls is 3 are independent.

- We simply need to decide if $P(B \cap C) = P(B) \cdot P(C)$.
- There are 36 equally-likely outcomes from rolling the two dice. In 6 of these B occurs, in 2 of these C occurs (1-2 or 2-1), but B and C can never occur together.
- Thus, $P(B) = 1/6$, $P(C) = 1/18$, and $P(B \cap C) = 0$.
- We see $P(B \cap C) = 0$ while $P(B) \cdot P(C) = 1/108$. Since these are not equal, the events B and C are not independent.

This finding of non-independence is also very reasonable for the simple reason that B and C are mutually exclusive: thus, knowing that one of them occurred provides very strong information about the other (namely, that it did not happen).

Independence, VI

Example: A single card is randomly dealt from a standard 52-card deck. If A is the event "the card is a seven" and B is the event "the card is not a king", determine whether A and B are independent.

Independence, VI

Example: A single card is randomly dealt from a standard 52-card deck. If A is the event "the card is a seven" and B is the event "the card is not a king", determine whether A and B are independent.

- There are 52 equally-likely outcomes from selecting one card. In 4 of these the card is a seven, in 48 of these the card is not a king, and also in 4 of these it is both a seven and not a king.
- Thus, $P(A) = 1/13$, $P(B) = 12/13$, and $P(A \cap B) = 1/13$.
- We see $P(A \cap B) = 1/13$ while $P(A) \cdot P(B) = 12/169$. Since these are not equal, the events A and B are not independent.

Try explaining why non-independence is reasonable here.

Independence, VII

<u>Example</u>: A single card is randomly dealt from a standard 52-card deck. If A is the event "the card is a seven" and C is the event "the card is a diamond or spade", determine whether A and C are independent.

Independence, VII

<u>Example</u>: A single card is randomly dealt from a standard 52-card deck. If A is the event "the card is a seven" and C is the event "the card is a diamond or spade", determine whether A and C are independent.

- In 4 outcomes the card is a seven, in 26 outcomes the card is a diamond or spade, and also in 2 outcomes it is both a seven and diamond or spade.
- Thus, $P(A) = 1/13$, $P(C) = 1/2$, and $P(A \cap C) = 1/26$.
- We see $P(A \cap C) = 1/26 = P(A) \cdot P(C)$, so the events A and C are independent.

Try explaining why independence is reasonable here.

Independence, VIII

Example: A single card is randomly dealt from a standard 52-card deck. If B is the event "the card is not a king" and C is the event "the card is a diamond or spade", determine whether B and C are independent.

Independence, VIII

<u>Example</u>: A single card is randomly dealt from a standard 52-card deck. If B is the event "the card is not a king" and C is the event "the card is a diamond or spade", determine whether B and C are independent.

- In 48 outcomes the card is not a king, in 26 outcomes the card is a diamond or spade, and in 24 of those it is also not a king.
- Thus, $P(B) = 12/13$, $P(C) = 1/2$, and $P(B \cap C) = 6/13$.
- We see $P(B \cap C) = 6/13 = P(B) \cdot P(C)$, so the events B and C are independent.

Note also that for these three events, A and C are independent, and B and C are independent, but A and B are not. (This says independence is not transitive.)

Independence of Complements, I

If A and B are independent events, then knowing that A occurs does not affect the probability that B occurs.

Under this interpretation, it is reasonable to expect that A^c and B should also be independent:

Proposition (Independence of Complements)

If A and B are independent events, then so are A^c and B.

Independence of Complements, I

If A and B are independent events, then knowing that A occurs does not affect the probability that B occurs.

Under this interpretation, it is reasonable to expect that A^c and B should also be independent:

Proposition (Independence of Complements)

If A and B are independent events, then so are A^c and B.

Proof:

- First observe $P(B) = P(A \cap B) + P(A^c \cap B)$, since the events $A \cap B$ and $A^c \cap B$ are mutually disjoint and have union B.
- If A and B are independent, then $P(A \cap B) = P(A)P(B)$, and therefore $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = [1 - P(A)] \cdot P(B) = P(A^c) \cdot P(B)$.
- This means A^c and B are also independent, as claimed.

Independence of Complements, II

Example: A single card is randomly dealt from a standard 52-card deck. If B is the event "the card is not a king" and C is the event "the card is a diamond or spade", verify that B^c and C are independent.

Independence of Complements, II

<u>Example</u>: A single card is randomly dealt from a standard 52-card deck. If B is the event “the card is not a king” and C is the event “the card is a diamond or spade”, verify that B^c and C are independent.

- Note that B^c is the event that the card is a king.
- From earlier, we have $P(B) = 12/13$ so $P(B^c) = 1/13$, and also $P(C) = 1/2$.
- Also, $B^c \cap C$ is the event that the card is a king that is a diamond or spade, so $P(B^c \cap C) = 2/52 = 1/26$.
- Thus, we indeed have $P(B^c \cap C) = 1/26 = P(B^c) \cdot P(C)$, so B^c and C are independent as claimed.

Collective Independence, I

We may also define independence of more than two events at once:

Definition

We say that the events $E_1, E_2, \ldots, E_n$ are collectively independent if $P(F_1 \cap F_2 \cap \cdots \cap F_k) = P(F_1) \cdot P(F_2) \cdot \cdots \cdot P(F_n)$ for any subset $F_1, F_2, \ldots, F_k$ of $E_1, E_2, \ldots, E_n$.

The intuitive idea is that the collection of events is independent whenever knowledge of whether some of the events have occurred does not affect the probability of any of the others.

Collective Independence, II

Example: A fair coin is flipped 3 times. If E_i is the event that the ith flip is heads for each $1 \leq i \leq 3$, show that E_1, E_2, E_3 are collectively independent.

Collective Independence, II

Example: A fair coin is flipped 3 times. If E_i is the event that the ith flip is heads for each $1 \leq i \leq 3$, show that E_1, E_2, E_3 are collectively independent.

- We have $P(E_1) = P(E_2) = P(E_3) = 1/2$.
- First, $P(E_1 \cap E_2) = 1/4 = P(E_1)P(E_2)$.
- Second, $P(E_1 \cap E_3) = 1/4 = P(E_1)P(E_3)$.
- Third, $P(E_2 \cap E_3) = 1/4 = P(E_2)P(E_3)$.
- Finally, $P(E_1 \cap E_2 \cap E_3) = 1/8 = P(E_1)P(E_2)P(E_3)$.
- Thus, these three events are collectively independent.

The collective independence of these events should be quite obvious, since each event concerns a different flip of the coin.

Collective Independence, III

For three events, it is already quite tedious to check collective independence (since we need to compute the probabilities of all their events and all possible intersections).

It is natural to wonder whether it is possible to skip some of these calculations (e.g., perhaps we can just check the intersection of each pair). In fact, this is not possible, as we show in the next example.

Collective Independence, IV

<u>Example</u>: A fair coin is flipped 3 times. If E_{ij} represents the event that the ith and jth flips are the same, show that each pair of E_{12}, E_{13}, E_{23} are independent but that they are not collectively independent.

Collective Independence, IV

<u>Example</u>: A fair coin is flipped 3 times. If E_{ij} represents the event that the ith and jth flips are the same, show that each pair of E_{12}, E_{13}, E_{23} are independent but that they are not collectively independent.

- We have $P(E_{12}) = P(E_{13}) = P(E_{23}) = 1/2$.
- Now, $E_{12} \cap E_{13}$ is the event where all 3 flips are the same, as are $E_{12} \cap E_{23}$ and $E_{13} \cap E_{23}$, so each of these pairwise intersections has probability $1/4$.
- Then we see $P(E_{12} \cap E_{13}) = 1/4 = P(E_{12}) \cdot P(E_{13})$ so these two are independent, as are the other two pairs.
- However, $E_{12} \cap E_{13} \cap E_{23}$ is also the event where all 3 flips are the same, so $P(E_{12} \cap E_{13} \cap E_{23}) = 1/4$, which is not equal to $P(E_{12}) \cdot P(E_{13}) \cdot P(E_{23}) = 1/8$.
- Thus, the events are not collectively independent.

Collective Independence, V

Our main application of collective independence is that it allows us to calculate probabilities of general sequences of independent events.

- Explicitly, suppose we have a sequence of collectively independent events $E_1, E_2, \ldots, E_n$.
- Then the probability of the intersection $P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdot \cdots \cdot P(E_n)$.
- So, if we can decompose a complicated event as the intersection of simpler independent events, we can find the probability of the intersection (namely, as the product above).

General Probabilities, I

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

1. She gets four hits during the game.
2. She gets a hit in her first two at-bats but not the other two.
3. She gets exactly two hits during the game.
4. She gets at least one hit in her last two at-bats.
5. She gets at least one hit during the game.

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4. She gets at least one hit in her last two at-bats.
5. She gets at least one hit during the game.

- Our sample space will be the $2^4 = 16$ possible outcomes of the four at-bats. However, not all the outcomes are equally likely.
- If E_i is the event of getting a hit on the ith at-bat, then $P(E_i) = 0.4$ and $P(E_i^c) = 1 - 0.4 = 0.6$.
- We can use the collective independence formula to find the probabilities of intersections of these events.

General Probabilities, II

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

1. She gets four hits during the game.

General Probabilities, II

<u>Example</u>: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

1. She gets four hits during the game.

- This event corresponds to the intersection $E_1 \cap E_2 \cap E_3 \cap E_4$.
- Since the four at-bats are collectively independent, the probability of getting a hit during all four at-bats is therefore $P(E_1 \cap E_2 \cap E_3 \cap E_4) = P(E_1) \cdot P(E_2) \cdot P(E_3) \cdot P(E_4) = 0.4^4 = 2.56\%$.

General Probabilities, III

<u>Example</u>: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

2. She gets a hit in her first two at-bats but not the other two.

General Probabilities, III

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

2. She gets a hit in her first two at-bats but not the other two.

- This event corresponds to the intersection $E_1 \cap E_2 \cap E_3^c \cap E_4^c$.
- Since the four at-bats are independent (and independent events also have independent complements), the probability of this event is therefore $P(E_1 \cap E_2 \cap E_3^c \cap E_4^c) = P(E_1) \cdot P(E_2) \cdot P(E_3^c) \cdot P(E_4^c) = 0.4 \cdot 0.4 \cdot 0.6 \cdot 0.6 = 5.76\%$.

General Probabilities, IV

<u>Example</u>: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

3. She gets exactly two hits during the game.

General Probabilities, IV

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

3. She gets exactly two hits during the game.

- First, we need to identify the various outcomes that make up this event.
- It is not hard to see that this event is the union of $\binom{4}{2} = 6$ possible events (one of which is event 2 from the previous slide), each of which has 2 hits and 2 non-hits in the 4 at-bats.
- By independence and the calculation for event (ii), each of these events has probability $0.4^2 \cdot 0.6^2$ and they are all mutually exclusive.
- Thus, the probability of their union is simply the sum of their individual probabilities, which is $6 \cdot 0.4^2 \cdot 0.6^2 = 34.56\%$.

General Probabilities, V

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

4. She gets at least one hit in her last two at-bats.

General Probabilities, V

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

4. She gets at least one hit in her last two at-bats.

- This event is the union of the two mutually exclusive events E_3 and $E_3^c \cap E_4$ (she either gets a hit in her third at-bat, or misses the third and makes the fourth).
- By independence, $P(E_3^c \cap E_4) = P(E_3^c) \cdot P(E_4) = 0.24$.
- Since $P(E_3) = 0.4$, we see that the probability of our event is the sum $P(E_3) + P(E_3^c \cap E_4) = 0.64 = 64\%$.
- Another approach would be to recognize that this event is the complement of not getting a hit in either of the last two at-bats (which is $E_3^c \cap E_4^c$, of probability $0.6 \cdot 0.6 = 0.36$).
- A third approach would be to list all 12 possible outcomes.

General Probabilities, VI

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

5. She gets at least one hit during the game.

General Probabilities, VI

Example: A baseball player's batting average is 0.4, meaning that she has a probability of 0.4 of getting a hit on any given at-bat, independently of any other at-bat. If she bats 4 times during a game, compute the probabilities of the following events:

5. She gets at least one hit during the game.

- One option would be to list all the possible outcomes that make up this event, and sum their respective probabilities.
- However, it is much quicker to recognize that this event is the complement of the event of not getting any hits in the game, which is $E_1^c \cap E_2^c \cap E_3^c \cap E_4^c$.
- So, by independence, we have $P(E_1^c \cap E_2^c \cap E_3^c \cap E_4^c) = P(E_1^c) \cdot P(E_2^c) \cdot P(E_3^c) \cdot P(E_4^c) = 0.6^4 = 0.1296$.
- Thus, the probability of getting at least one hit in the game is $1 - 0.1296 = 0.8704 = 87.04\%$.

Probability Medley, I

Example: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

1. Both rolls are sixes.
2. At least one roll is a six.
3. The sum of the rolls is 10.
4. The two rolls are equal.

Probability Medley, I

<u>Example</u>: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

1. Both rolls are sixes.
2. At least one roll is a six.
3. The sum of the rolls is 10.
4. The two rolls are equal.

- Our sample space will be the $6^2 = 36$ possible pairs of rolls. However, not all the outcomes are equally likely.
- The given information tells us that for each die, the probability of a 1, 2, 3, 4, or 5 is $1/7$ while the probability of a 6 is $2/7$.
- Now we use the collective independence formula to find the probabilities of the listed events.
- We will write A_i for the event of rolling i with die A and B_i for rolling i with die B.

Probability Medley, II

<u>Example</u>: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

1. Both rolls are sixes.

Probability Medley, II

Example: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

1. Both rolls are sixes.

- This event is the intersection $A_6 \cap B_6$.
- Since $P(A_6) = P(B_6) = 2/7$, and the rolls are independent, we have $P(A_6 \cap B_6) = P(A_6) \cdot P(B_6) = 4/49$.

Probability Medley, III

Example: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

2. At least one roll is a six.

Probability Medley, III

<u>Example</u>: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

2. At least one roll is a six.

- This event is the union $A_6 \cup B_6$. By the union-intersection formula, we have $P(A_6 \cup B_6) = P(A_6) + P(B_6) - P(A_6 \cap B_6) = 2/7 + 2/7 - 4/49 = 24/49$.
- Alternatively, notice that this event is the complement of not rolling a 6 with either die, which is $A_6^c \cap B_6^c$.
- Since $P(A_6^c \cap B_6^c) = P(A_6^c) \cdot P(B_6^c) = (5/7) \cdot (5/7) = 25/49$, the original event has probability $1 - 25/49 = 24/49$.

Probability Medley, IV

Example: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

3. The sum of the rolls is 10.

Probability Medley, IV

Example: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

3. The sum of the rolls is 10.

- There are three possible outcomes that make up this event: (6,4), (5,5), and (4,6), which correspond to the intersections $A_6 \cap B_4$, $A_5 \cap B_5$, and $A_4 \cap B_6$.
- The probabilities of these three intersections are $P(A_6 \cap B_4) = (2/7) \cdot (1/7) = 2/49$, $P(A_5 \cap B_5) = (1/7) \cdot (1/7) = 1/49$, and $P(A_4 \cap B_6) = (1/7) \cdot (2/7) = 2/49$.
- Therefore, the probability of this event is simply the sum $(2/49) + (1/49) + (2/49) = 5/49$.

Probability Medley, V

Example: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

4. The two rolls are equal.

Probability Medley, V

Example: Two 6-sided dice are rolled. The dice are weighted so that rolling a 6 is twice as likely as any one of the other 5 equally likely outcomes. Find the probabilities of the following events:

4. The two rolls are equal.

- There are six outcomes that make up this event: (1,1), (2,2), (3,3), (4,4), (5,5), and (6,6).
- Like in the previous parts, we can find the probabilities of these outcomes by multiplying the appropriate individual die probabilities. We see that the first five each have probability $(1/7) \cdot (1/7) = 1/49$ while the last has probability $(2/7) \cdot (2/7) = 4/49$.
- Therefore, the probability of this event is $5 \cdot (1/49) + (4/49) = 9/49$.

Summary

We discussed more properties of conditional probabilities.

We introduced the notion of independence of events and discussed some properties of independence.

We discussed how to compute more general probabilities using independence.

Next lecture: Computing general probabilities.