Math 3081 (Probability and Statistics)

Lecture #2 of 27 $\sim$ July 7th, 2021

Counting Principles

- Permutations
- Combinations
- Sample Spaces and Events

This material represents §1.2.2-1.3.1 from the course notes.

Recall

Last time, we introduced some properties of sets, and we introduced some counting principles:

Principle (Addition Principle)

When choosing among n disjoint options labeled 1 through n, if option i has a_i possible outcomes for each $1 \leq i \leq n$, then the total number of possible outcomes is $a_1 + a_2 + \cdots + a_n$.

Principle (Multiplication Principle)

When making a sequence of n independent choices, if step i has b_i possible outcomes for each $1 \leq i \leq n$, then the total number of possible collections of choices is $b_1 \cdot b_2 \cdot \cdots \cdot b_n$.

Today, we will tackle some more complex counting problems.

Permutations, I

Certain problem types involving rearrangements of distinct objects, known as permutations, arise frequently in counting problems.

Example: Determine the number of permutations (i.e., ways to rearrange) the six letters ABCDEF.

Permutations, I

Certain problem types involving rearrangements of distinct objects, known as permutations, arise frequently in counting problems.

Example: Determine the number of permutations (i.e., ways to rearrange) the six letters ABCDEF.

- There are 6 letters to be arranged into 6 locations.
- For the first letter, there are 6 choices (any of ABCDEF).
- For the second letter, there are only 5 choices (any letter except the one we have already chosen).
- For the third letter, there are only 4 choices (any letter except the first two).
- Continuing in this way, we see that there are 3 choices for the fourth letter, 2 choices for the fifth letter, and only 1 choice for the last letter.
- By the multiplication principle, the total number of permutations is therefore $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

Permutations, II

Example: A new company logo has four design elements, which must all be different colors chosen from red, orange, yellow, green, blue, purple, and pink. How many different logos are possible?

Permutations, II

Example: A new company logo has four design elements, which must all be different colors chosen from red, orange, yellow, green, blue, purple, and pink. How many different logos are possible?

- There are 7 possible colors.
- The first design element has 7 possible colors.
- The second element has 6 possible colors (any of the 7 except the one already used).
- In the same way, the third element has 5 possible colors, and the fourth has 4 possible colors.
- Thus, the total number of logos is $7 \cdot 6 \cdot 5 \cdot 4 = 840$.

Permutations, III

Example: A basketball team has 18 roster players, and must choose a starting lineup, which consists of a center, power forward, small forward, point guard, and shooting guard. If each player can play all 5 positions, how many starting lineups are possible?

Permutations, III

Example: A basketball team has 18 roster players, and must choose a starting lineup, which consists of a center, power forward, small forward, point guard, and shooting guard. If each player can play all 5 positions, how many starting lineups are possible?

- There are 18 possible choices for the center.
- Once the center is chosen, there are 17 remaining possibilities for the power forward.
- Once these two are chosen, there are 16 choices for the small forward, and then 15 for the point guard, and finally 14 for the shooting guard.
- Thus, the total number of starting lineups is $18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 = 1\,028\,160$.

Permutations, IV

These three examples are all computing permutations, in which we choose k distinct items from a list of n possibilities, and where the order of our choices matters. We can give a general formula for this type of problem using factorials:

Definition

If n is a positive integer, we define the number $n!$ (read "n factorial") as $n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$, the product of the positive integers from 1 to n inclusive. We also set $0! = 1$.

- Some small values are $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, and $6! = 720$.
- The factorial function grows very fast: to 4 significant figures, we have $10! = 3.629 \cdot 10^6$, $100! = 9.333 \cdot 10^{157}$, and $1000! = 4.024 \cdot 10^{2567}$.

Permutations, V

Now we can give a formula for counting permutations:

Proposition (Counting Permutations)

The number of ways of choosing k ordered items from a list of n distinct possibilities (where the order of the k items matters) is equal to $\frac{n!}{(n-k)!} = n \cdot (n-1) \cdot \dots \cdot (n-k+1)$. *In particular, the number of ways of rearranging n distinct items is $n!$.*

Permutations, V

Now we can give a formula for counting permutations:

Proposition (Counting Permutations)

The number of ways of choosing k ordered items from a list of n distinct possibilities (where the order of the k items matters) is equal to $\frac{n!}{(n-k)!} = n \cdot (n-1) \cdot \dots \cdot (n-k+1)$. *In particular, the number of ways of rearranging n distinct items is $n!$.*

<u>Proof</u>:

- There are n possibilities for the first item, $n-1$ for the second item (any possibility but the one already chosen), $n-2$ for the third item (any possibility but the two already chosen), ..., and $n-k+1$ possibilities for the kth item.
- This yields a total number of possibilities of $n \cdot (n-1) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}$ as claimed.

Permutations, VI

<u>Example</u>: A sports league has 31 teams in total. How many ways are there to choose 16 teams that make the playoffs, assuming that the ranking of the playoff teams matters and there are no ties?

Permutations, VI

Example: A sports league has 31 teams in total. How many ways are there to choose 16 teams that make the playoffs, assuming that the ranking of the playoff teams matters and there are no ties?

- We are choosing $k = 16$ teams from a list of $n = 31$, where the order matters.
- Thus, from the permutation counting formula, the total number of choices is $\boxed{\frac{31!}{15!}} = 31 \cdot 30 \cdot \cdots \cdot 16$.
- This is very big to compute by hand, but a computer can quickly evaluate it: $393\,008\,709\,555\,221\,760\,000 \approx 3.930 \cdot 10^{20}$.

Combinations, I

Another variation on permutations also shows up often; namely, in which the order of the list of the k items we choose from the list of n does not matter. Such selections are known as combinations:

Proposition (Counting Combinations)

The number of ways of choosing k unordered items from a list of n distinct possibilities is $\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot\cdots\cdot(n-k+1)}{k\cdot(k-1)\cdot\cdots\cdot 1}$.

Combinations, I

Another variation on permutations also shows up often; namely, in which the order of the list of the k items we choose from the list of n does not matter. Such selections are known as combinations:

Proposition (Counting Combinations)

The number of ways of choosing k unordered items from a list of n distinct possibilities is $\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot\cdots\cdot(n-k+1)}{k\cdot(k-1)\cdot\cdots\cdot 1}$.

Proof:

- If instead we want the k items to be ordered, we are counting permutations, of which there are $\frac{n!}{(n-k)!}$.
- Now simply observe that for any unordered list, there are $k!$ ways to rearrange the k elements on the list.
- Thus we have counted each unordered list $k!$ times, so the number of unordered lists is $\frac{1}{k!} \cdot \frac{n!}{(n-k)!} = \frac{n!}{k!(n-k)!}$.

Combinations, II

Some remarks:

- The symbols $\binom{n}{k}$ and ${}_nC_k$ are both typically read as "n choose k". We will exclusively use the notation $\binom{n}{k}$.
- The numbers $\binom{n}{k}$ are called <u>binomial coefficients</u> because they arise as coefficients of binomial expansions.
- Specifically, in the expansion of $(x+y)^n$, the coefficient of $x^k y^{n-k}$ is equal to $\binom{n}{k}$.
- For example, we can compute that $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, and the middle term is indeed equal to $\binom{4}{2} = \frac{4!}{2!2!} = 6$.
- Binomial coefficients show up in many different places, and satisfy numerous identities, such as the "reflection identity" $\binom{n}{k} = \binom{n}{n-k}$ along with the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Combinations, III

The binomial coefficients can be organized into a famous array called Pascal's triangle:

Row 0							1						
Row 1						1		1					
Row 2					1		2		1				
Row 3				1		3		3		1			
Row 4			1		4		6		4		1		
Row 5		1		5		10		10		5		1	
Row 6	1		6		15		20		15		6		1

The entries in row n are the values $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{2}$, ... , $\binom{n}{n}$.
Each entry in row $n+1$ is obtained by summing the two entries above it in row n.

Combinations, IV

In general, expanding the products of factorials is not the most efficient way to evaluate binomial coefficients.

- Instead, the formula $\binom{n}{k} = \dfrac{n \cdot (n-1) \cdot \cdots \cdot (n-k+1)}{k \cdot (k-1) \cdot \cdots \cdot 1}$ is typically the most efficient.
- For example, computing $\binom{13}{4}$ as $\dfrac{13!}{4!9!}$ requires computing both 13! and 4!9!, and then evaluating the quotient, which is rather painful to do by hand.
- On the other hand, the formula above gives $\binom{13}{4} = \dfrac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 11 \cdot 5 = 715$, which is easy to evaluate by hand.

Combinations, V

Example: How many 3-element subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are there?

Combinations, V

Example: How many 3-element subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are there?

- Since subsets are not ordered, we are simply counting the number of ways to choose 3 unordered elements from the given set of 9.
- From our discussion of combinations, the number of such subsets is $\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$.

Combinations, VI

Example: At a conference with 30 mathematicians, every pair of attendees shares a socially-distanced virtual handshake once. How many total virtual handshakes occur?

Combinations, VI

Example: At a conference with 30 mathematicians, every pair of attendees shares a socially-distanced virtual handshake once. How many total virtual handshakes occur?

- Since pairs of people are not ordered, we are counting the number of ways to choose 2 attendees from a total of 30, which is $\binom{30}{2} = \frac{30 \cdot 29}{2 \cdot 1} = 435$.

Combinations, VII

<u>Example</u>: A pizza parlor offers 14 different possible toppings on a pizza. A pizza may have from 0 up to 4 different toppings. How many different pizza topping combinations are possible?

Combinations, VII

Example: A pizza parlor offers 14 different possible toppings on a pizza. A pizza may have from 0 up to 4 different toppings. How many different pizza topping combinations are possible?

- In general, there are $\binom{14}{k}$ possible pizzas that have exactly k toppings, since toppings cannot be repeated and the order does not matter.
- Thus, the number of pizzas with at most 4 toppings is $\binom{14}{0} + \binom{14}{1} + \binom{14}{2} + \binom{14}{3} + \binom{14}{4}$ $= 1 + 14 + 91 + 364 + 1001 = 1471$.

Combinations, VIII

Example: Determine the number of different 5-card hands that can be dealt from a standard 52-card deck. (The order of cards in a hand does not matter.)

Combinations, VIII

Example: Determine the number of different 5-card hands that can be dealt from a standard 52-card deck. (The order of cards in a hand does not matter.)

- Since the order does not matter, we are counting the number of combinations of 5 cards selected from the deck of 52.
- Thus, there are $\binom{52}{5} = 2\,598\,960$ possible 5-card hands.

Combinations, IX

Example: Determine the number of different 5-card flush hands, consisting of 5 cards of the same suit, that can be dealt from a standard 52-card deck.

Combinations, IX

Example: Determine the number of different 5-card flush hands, consisting of 5 cards of the same suit, that can be dealt from a standard 52-card deck.

- There are 4 possible choices of suit. Once we make this selection, we then select 5 unordered cards from the 13 cards in that suit.
- There are $\binom{13}{5} = 1287$ ways to choose these 5 cards.
- Thus, by the multiplication principle, in total there are $4 \cdot \binom{13}{5} = 5148$ possible 5-card flushes.

Combinations, X (Marks The Spot)

Example: Determine the number of different full-house hands, consisting of three cards of one rank and a pair of cards in another rank, that can be dealt from a standard 52-card deck.

Combinations, X (Marks The Spot)

Example: Determine the number of different full-house hands, consisting of three cards of one rank and a pair of cards in another rank, that can be dealt from a standard 52-card deck.

- Note that there are 13 possible card ranks, and 4 cards of each rank.
- First, there are 13 ways to choose the rank of the 3-of-a-kind, and then there are 12 ways to choose the rank of the pair.
- Once we have chosen the ranks, there are $\binom{4}{3} = 4$ ways to choose the three cards forming the 3-of-a-kind, and there are $\binom{4}{2} = 6$ ways to choose the two cards forming the pair.
- Thus, in total there are $13 \cdot 12 \cdot 4 \cdot 6 = 3744$ possible full houses.

Combinations, XI

Example: Determine the number of possible ways of permuting the letters in the word MISSISSIPPI.

Combinations, XI

Example: Determine the number of possible ways of permuting the letters in the word MISSISSIPPI.

- Since there are 11 letters, it might seem as if there are 11! permutations of the letters. However, not all permutations yield different words: for example, if we swap two of the Ss, the resulting words are the same.
- There are 4 Ss, 4 Is, 2 Ps, and 1 M, which we arrange in order.
- First, place the 4 Ss: since there are 11 possible locations, there are $\binom{11}{4}$ ways to place them.
- Next, place the 4 Is: there are 7 remaining locations, so there are $\binom{7}{4}$ ways to place them.
- In the same way, there are $\binom{3}{2}$ choices for the 2 Ps, and then just $\binom{1}{1}$ choice for the M.
- In total, there are $\binom{11}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} \cdot \binom{1}{1} = 330 \cdot 35 \cdot 3 = 34650$ possible permutations of MISSISSIPPI.

Combinations, XII

Example: Determine the number of possible ways of permuting the letters in the word BOSTONIANS that contain the word BOOS.

Combinations, XII

Example: Determine the number of possible ways of permuting the letters in the word BOSTONIANS that contain the word BOOS.

- The number of such permutations is the number of permutations of the six letters T, N, I, A, N, S and the string BOOS (which we can think of as being a single string).
- There are 2 Ns, and 1 each of T, I, A, S, and BOOS to arrange.
- First, we place the 2 Ns: since there are 7 possible locations, there are $\binom{7}{2}$ ways to place them. The remaining 5 strings can be permuted arbitrarily, so there are 5! ways to arrange them.
- In total, there are $\binom{7}{2} \cdot 5! = 42 \cdot 120 = 2520$ ways of permuting the letters.

Counting Medley, I

Example: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given condition:

1. No conditions.
2. The string starts with A.
3. The string has no repeated letters.
4. The string has at least one repeated letter.
5. The string does not contain the letter B.
6. The string contains at least one letter B.
7. Each letter is different from the one before it.
8. The string has no repeated letters and is in alphabetical order.

We will do these one at a time.

Counting Medley, II

Example: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

1. No conditions.

Counting Medley, II

Example: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

1. No conditions.

- Each letter in the string has 8 possible choices, so there are $8^4 = 4096$ such strings.

2. The string starts with A.

Counting Medley, II

<u>Example</u>: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

1. No conditions.

- Each letter in the string has 8 possible choices, so there are $8^4 = 4096$ such strings.

2. The string starts with A.

- The first letter has 1 possible choice and the remaining 3 have 8 possible choices, so there are $8^3 = 512$ such strings.

Counting Medley, III

<u>Example</u>: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

3. The string has no repeated letters.

Counting Medley, III

Example: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

3. The string has no repeated letters.
   - Because order matters, we are counting permutations of 4 letters taken from a selection of 8.
   - Thus, there are $8!/4! = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$ such strings.
4. The string has at least one repeated letter.

Counting Medley, III

<u>Example</u>: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

3. The string has no repeated letters.
   - Because order matters, we are counting permutations of 4 letters taken from a selection of 8.
   - Thus, there are $8!/4! = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$ such strings.
4. The string has at least one repeated letter.
   - This might seem like it would be very hard to count, because there are many ways a string could have a repeated letter (a double letter, a triple letter, two double letters, etc.).
   - But in fact, this set of strings is simply the complement of the ones we just counted above.
   - So, the number of strings with at least one repeated letter is $4096 - 1680 = 2416$.

Counting Medley, IV

Example: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

5. The string does not contain the letter B.

Counting Medley, IV

Example: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

5. The string does not contain the letter B.
   - There are now 7 possible choices for each of the 4 letters in the string, so the total is simply $7^4 = 2401$.
6. The string contains at least one letter B.

Counting Medley, IV

<u>Example</u>: Consider all of the 4-letter strings that can be made from the letters ABCDEFGH. Determine the number of strings satisfying the given conditions:

5. The string does not contain the letter B.

- There are now 7 possible choices for each of the 4 letters in the string, so the total is simply $7^4 = 2401$.

6. The string contains at least one letter B.

- Like in part 4, this set of strings is simply the complement of the ones we just counted above.
- So, the number of strings with at least one letter B is $4096 - 2401 = 1695$.

Counting Medley, V

Example: 4-letter strings from ABCDEFGH:

7. Each letter is different from the one before it.

Counting Medley, V

Example: 4-letter strings from ABCDEFGH:

7. Each letter is different from the one before it.

- There are 8 choices for the first letter. Once we choose it, there are 7 choices for the second letter (anything but the one chosen).
- Once we choose the second letter, there are 7 choices for the third letter (anything but the second letter).
- For the same reason, there are 7 choices for the fourth letter. Hence the total is $8 \cdot 7 \cdot 7 \cdot 7 = 2744$.

8. The string has no repeated letters and is in alphabetical order.

Counting Medley, V

Example: 4-letter strings from ABCDEFGH:

7. Each letter is different from the one before it.
   - There are 8 choices for the first letter. Once we choose it, there are 7 choices for the second letter (anything but the one chosen).
   - Once we choose the second letter, there are 7 choices for the third letter (anything but the second letter).
   - For the same reason, there are 7 choices for the fourth letter. Hence the total is $8 \cdot 7 \cdot 7 \cdot 7 = 2744$.
8. The string has no repeated letters and is in alphabetical order.
   - Given any set of letters, there is only one way to put them in alphabetical order. So in fact, we are just choosing subsets of 4 distinct letters from our set of 8 (i.e., combinations).
   - Thus, there are $\binom{8}{4} = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$ such strings.

Sample Spaces and Events, I

Our goal now is to use some of these properties of sets and counting to develop the notion of probability. Today, we will take a fairly abstract viewpoint based on sets, and then next time we will specialize to more concrete settings, which will involve applying counting techniques.

Sample Spaces and Events, I

Our goal now is to use some of these properties of sets and counting to develop the notion of probability. Today, we will take a fairly abstract viewpoint based on sets, and then next time we will specialize to more concrete settings, which will involve applying counting techniques.

The fundamental idea of "probability" arises from performing an experiment or observation, and tabulating how often particular outcomes arise. We need to make this a bit more precise:

"Definition"

For any experiment or observation, the set of possible outcomes is called the sample space, and an event is a subset of the sample space.

Sample Spaces and Events, II

Example: Consider the experiment of "rolling a standard 6-sided die once".

- There are 6 possible outcomes to this experiment, namely, rolling a 1, a 2, a 3, a 4, a 5, or a 6, so the sample space is the set $S = \{1, 2, 3, 4, 5, 6\}$.
- One event is "rolling a 3", which would correspond to the subset $\{3\}$.
- Another event is "rolling an even number", which would correspond to the subset $\{2, 4, 6\}$.
- A third event is "rolling a number bigger than 2", which would correspond to the subset $\{3, 4, 5, 6\}$.
- A fourth event is "rolling a negative number", which would correspond to the empty subset $\emptyset = \{\}$ because there are no outcomes in the sample space that make this event occur.

Sample Spaces and Events, III

Example: Consider the experiment of "flipping a coin once".

- There are 2 possible outcomes to this experiment: heads and tails. Thus, the sample space is the set $S = \{\text{heads, tails}\}$, which we typically abbreviate as $S = \{H, T\}$.
- One event is "obtaining heads", corresponding to the subset $\{H\}$.
- Another event is "obtaining tails", corresponding to the subset $\{T\}$.

Sample Spaces and Events, IV

Example: Consider the experiment of "flipping a coin four times".

- By the multiplication principle, there are $2^4 = 16$ possible outcomes to this experiment (namely, the 16 possible strings of 4 characters each of which is either H or T).
- One event is "exactly one head is obtained", corresponding to the subset $\{HTTT, THTT, TTHT, TTTH\}$.
- Another event is "the first three flips are tails", corresponding to the subset $\{TTTH, TTTT\}$.

Sample Spaces and Events, V

Example: Consider the experiment of "drawing one card from a standard 52-card deck".

- Since there are 52 possible cards, there are 52 outcomes in the sample space. Some examples include "the four of spades", "the ace of clubs", and "the jack of hearts".
- We could abbreviate each card by its numerical label (A,2,3,4,5,6,7,8,9,10,J,Q,K) and its suit ($\heartsuit$, $\clubsuit$, $\diamondsuit$, $\spadesuit$).
- One event is "a king is drawn", corresponding to the subset $\{K\heartsuit, K\clubsuit, K\diamondsuit, K\spadesuit\}$.
- Another event is "a spade is drawn", corresponding to $\{A\spadesuit, 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit\}$.
- A third event is "a red jack is drawn", corresponding to $\{J\heartsuit, J\diamondsuit\}$.

Sample Spaces and Events, VI

Example: Consider the experiment of "measuring the lifetime of a refrigerator in years".

- There are many possible outcomes to this experiment, including 0, 5, 28, 3.2, and 100.
- The sample space would (at least in principle) be the set of nonnegative real numbers: $S = [0, \infty)$.
- One event is "the refrigerator stops working after at most 3 years", corresponding to the subset $S = [0, 3]$.
- Another event is "the refrigerator works for at least 6 years", corresponding to the subset $S = [6, \infty)$.
- A third event is "the refrigerator is broken when it arrives", corresponding to the subset $S = \{0\}$.

Sample Spaces and Events, VII

Example: Consider the experiment of "shuffling a standard 52-card deck".

- Since there are 52 possible cards, there are 52! outcomes in the sample space, one for each possible permutation of the cards.
- The sample space S is the (quite large!) set of all 52! of these sequences.
- One event is "the first card is an ace of clubs", while another event is "the entire deck alternates red-black-red-black...-red-black".
- It would be quite infeasible to write out the exact subsets corresponding to these events (it is left as an exercise for you to verify that their cardinalities are $51!$ and $26! \cdot 26!$ respectively), but in principle we could list them all.

Sample Spaces and Events, VIII

Example: Consider the experiment of "measuring the temperature in degrees Fahrenheit outside".

- There are many possible outcomes to this experiment, including 28°, 87.4°, and 120°.
- Depending on how accurately the temperature is measured, and what the possible outside temperatures are, the sample space could be very large or even infinite.
- If the temperature is measured to the nearest whole degree, and it is never colder than 0° nor hotter than 120°, then the sample space would be $S = \{0^\circ, 1^\circ, 2^\circ, \dots, 120^\circ\}$.
- One event is "the temperature is above freezing", corresponding to the subset $\{33^\circ, 34^\circ, \dots, 120^\circ\}$.
- Another event is "the temperature is a multiple of 10°", corresponding to the subset $\{0^\circ, 10^\circ, 20^\circ, \dots, 120^\circ\}$.

Operations on Events, I

Since we view events as subsets of the sample space, we can perform the same operations on events as we can on sets.

Definition

If A and B are events in a sample space S, we define the union $A \cup B$ to be the event corresponding to the union $A \cup B$. We also define the intersection $A \cap B$ and the complement A^c analogously.

In the case where the events A and B have $A \cap B = \emptyset$, we say that A and B are mutually exclusive, since they cannot both occur at the same time.

Operations on Events, II

Example:

- Let S be the sample space $S = \{1, 2, 3, 4, 5, 6\}$ obtained by rolling a standard 6-sided die once, with $A = \{2, 4, 6\}$ the event of rolling an even number and $B = \{3, 4, 5, 6\}$ the event of rolling a number larger than 2.
- Then the complement $A^c = \{1, 3, 5\}$ is the event of not rolling an even number (i.e., rolling an odd number), and the complement $B^c = \{1, 2\}$ is the event of not rolling a number larger than 2.
- Also, the union $A \cup B = \{2, 3, 4, 5, 6\}$ is the event of rolling a number that is even or larger than 2, while the intersection $A \cap B = \{4, 6\}$ is the event of rolling a number that is even and larger than 2.

Motivation for Probability, I

We have laid out the general framework for events and sample spaces, so now we can begin studying probabilities of events, which quantify how likely it is that a particular event will occur.

Rather than simply giving the definition, here is a bit of motivation for how this should align with your intuition:

- If we have an experiment with corresponding sample space S, and E is an event (which we consider as being a subset of S), we would like to define the probability of E, written $P(E)$, to be the frequency with which E occurs if we repeat the experiment many times independently.

Motivation for Probability, II

- Specifically, if we repeat the experiment n times and the event occurs e_n times, then the relative frequency that E occurs is the ratio e_n/n. If we let n grow very large (more formally, if we take the limit as $n \to \infty$) then the ratios e_n/n should approach a fixed number, which we call the probability of the event E.
- Since for each n we have $0 \leq e_n \leq n$, and thus $0 \leq e_n/n \leq 1$, we see that the limit of the ratios e_n/n must be in the closed interval $[0, 1]$. This tells us that $P(E)$ should always lie in this interval.
- For the event S (consisting of the entire sample space), we clearly have $P(S) = 1$, because if we perform the experiment n times, the event S always occurs n times, so $e_n = n$ for every n.

Motivation for Probability, III

- Also, if E_1 and E_2 are mutually exclusive events, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$: this follows by observing that because both events cannot occur simultaneously, then the total number of times E_1 or E_2 occurs in n experiments is equal to the total number of times E_1 occurs plus the total number of times E_2 occurs.
- In particular, if $S = \{s_1, s_2, \ldots, s_k\}$ is a finite sample space, then by applying the above observation repeatedly, we can see that $P(\{s_1\}) + P(\{s_2\}) + \cdots + P(\{s_k\}) = P(\{s_1, s_2, \ldots, s_k\}) = P(S) = 1$.
- This means that the sum of all the probabilities of the events in the sample space is equal to 1.

Motivation for Probability, IV

To summarize, if probabilities behave in accordance with the properties we just worked out using this intuitive formulation, they should have the following properties:

1. The probability of any event should represent the proportion of times it occurs. In particular, it should always be between 0 and 1 (inclusive).
2. The probability of the entire sample space S should equal 1.
3. If E_1 and E_2 are mutually exclusive events, then $P(E_1 \cup E_2)$ should equal $P(E_1) + P(E_2)$.

The clever idea, first proposed by Kolmogorov, is that we should actually turn this logic on its head, and *define* probabilities of events using these properties. We will go through this next time.

Summary

We introduced factorials and the binomial coefficients, and used them to count permutations and combinations.

We discussed a number of examples of using permutations and combinations to count things.

We introduced events and sample spaces and gave some motivation for probability.

Next lecture: Calculating probabilities.