Math 3081 (Probability and Statistics)

Lecture #1 of 27 $\sim$ July 6, 2021

Introduction + Sets and Counting

- Welcome to Math 3081 + Course Logistics
- Sets, Subsets, Cardinality
- Unions, Intersections, Complements, Venn Diagrams
- Counting: Addition and Multiplication Principles

This material represents §1.1-§1.2.1 from the course notes.

Welcome!

Welcome to Math 3081 (Probability and Statistics)! Here are some course-related locations to bookmark:

- The course webpage is here: `https://web.northeastern.edu/dummit/teaching_su21_3081.html` . Most course-related information is posted there.
- Homework assignments are in WeBWorK via Rederly: `https://app.rederly.com/` .
- Course-related discussion will be done via Piazza: `https://piazza.com/class/kqb79ko8yki2d2`
- Exams will be distributed and collected via Canvas: `https://canvas.northeastern.edu/`

Course Topics: Probability and Statistics

As you might expect based on the title, we will be covering probability and statistics in this course. More specifically:

- **Probability**: sets, counting, probability distributions, conditional probability, independence, applications of probability, discrete and continuous random variables, joint distributions, expected value / variance / standard deviation, the binomial / Poisson / exponential / normal distributions, the central limit theorem, modeling applications.
- **Statistics**: parameter estimation, maximum likelihood estimates, interval estimation, hypothesis testing, p-values, z scores and z testing, errors and misuses of hypothesis tests, Student's and Welch's t tests, the χ^2 tests for independence and goodness-of-fit, and statistical methodology.

The term is split evenly between these two clusters of topics.

Lectures

The course lectures will be conducted via Zoom. All lectures are recorded for later viewing. For security reasons (since these lecture slides are posted publicly) the links to upcoming and past lectures are only available via the Canvas page or via the Piazza page.

- Section 1 meets MTWR from 1:30pm-3:10pm Eastern time.
- Section 2 meets MTWR from 9:50am-11:30am Eastern time.

Attendance is highly encouraged but not required. Since I am teaching both sections of 3081 this term, for your convenience you may attend either lecture (they will cover the same material at the same pace).

Grades and Exams, I

The assignments in this class consist of weekly WeBWorK (homework) and biweekly exams. More specifically:

- There will be 7 WeBWorK assignments, one each week. Every problem on every assignment will be counted, and your score is [total points scored]/[total problems assigned].
- There will be 4 exams: three midterms and a final. Exams will be distributed via Canvas and will have the same format and length as if they were being given in class.
- You will sign up for a fixed time window in which to take your exam and submit scans of your responses. If you miss an exam for any reason, you will receive a zero. Makeup exams will not be given. The final can replace a missed midterm.

Grades and Exams, II

There are four exams in this course: three 60-min midterm exams and a final exam.

- The three midterms will be held on Sat Jul 17, Sat Jul 31, and Sat Aug 14 (every two weeks starting in week 2). You will have an option of at least four possible time windows during which to take the exam, spread throughout the day.
- The final exam will be held on Mon Aug 23 or Tue Aug 24 and will have at least five time options spread across the two days.
- There are two options for the final: a 2-hour comprehensive final, or a 45-minute short final. You will decide which one to take after the last class day.
- The comprehensive final covers the whole semester, while the short final only covers the material after midterm 3.

Grades and Exams, III

Your course grade consists of 15% WeBWorK (homework) and 85% exams.

- If you take the comprehensive final, your exam score is the maximum of ($3\times 20\%$ each midterm $+$ 25% final) and ($2\times 25\%$ best two midterms $+$ 35% final).
- If you take the short final, your exam score is ($3\times 23\%$ each midterm $+$ 16% final).
- Thus, with the short final, all of the midterms count and you cannot drop any scores, so this option is only recommended if you have done well on all of the other exams.
- On the other hand, if you take the comprehensive final, you have the opportunity to make up for a low or missing midterm grade (since the lowest midterm can be dropped).

Grades and Exams, IV

The letter grades in the course are not assigned via a fixed scale (it is determined by the distribution at the end of the term). However, there are lower bounds:

- A raw average of 92% will be at least an A.
- A raw average of 90% will be at least an A-.
- A raw average of 88% will be at least a B+.
- A raw average of 82% will be at least a B.
- A raw average of 80% will be at least a B-.
- A raw average of 78% will be at least a C+.
- A raw average of 72% will be at least a C.
- A raw average of 60% will guarantee a passing grade.

Approximate grade correspondences will be provided with the midterm exam scores so that you can track your progress.

WeBWorK, I

There will be 7 WeBWorK assignments, one each week, typically due at 5am Eastern on Fridays. (The first assignment is instead due **this Sunday** at 5am, since it is the first week.)

- WeBWorK is an open-source electronic homework system designed specifically for mathematics and statistics courses. The main advantage (for you) is that it is free for students.
- Each assignment will have approximately 20 problems. All problems are worth 1 point; many problems have multiple parts, each of which is worth some fraction of 1 point.
- You are encouraged to consider the homeworks as being due "Thursday evening".

You are allowed a 12-hour grace period after the official due time of 5am, in which you may continue working on problems for 50% credit. Extensions can only be given in absolute emergencies.

WeBWorK, II

To access WeBWorK, go to `https://app.rederly.com/` and log in with your Northeastern email address and password.

- Your WeBWorK username is your Northeastern email address. (You must use your Northeastern email address!)
- To set up your account password, check your Canvas messages for the invitation link to create your password.

Try logging into WeBWorK now to check that things work properly, if you haven't already done so.

WeBWorK, III

How to use WeBWorK as wisely as possible:

- Consider the homeworks as being due "Thursday evening". (WeBWorK 1 is due two days later since it is the first week.)
- Look over the problems as soon as the set is open so you know what material is covered, and then spread out your work on the problems over several days. Do not fall into the trap of only starting the assignment the evening before it is due!
- If you are stuck on a problem, move to another one and then come back. If you're still stuck, ask for help on Piazza, during office hours, or using WeBWorK's "Email Instructor" button.
- Make sure to do as much of every assignment as possible, even if you cannot completely finish it.

Because of the compressed nature of the course, the midterm exams on Saturdays will cover the material on the WeBWorK assignment due immediately before.

Piazza

Math 3081 will use Piazza as a course discussion forum: `https://piazza.com/class/kqb79ko8yki2d2`. The Piazza page is open only by invitation to 3081 students and is linked to Canvas. Check your email for the invitation.

- Links to the recordings of the past lectures will be hosted on the Piazza page, as well as the link to upcoming lectures and problem sessions.
- You are encouraged to conduct discussions about the course and associated topics on Piazza. In particular, you are welcome to ask questions about the WeBWorK problems. If you do, it is helpful to others if you include a snapshot of your version of the problem.

Try visiting the Piazza page now, if you haven't already done so.

Problem Sessions + Office Hours

The TA will hold a weekly problem session on Thursdays from noon to 1pm via Zoom. It is devoted to going over problems on the recent course material similar to the problems on the upcoming WeBWorK, and is recorded for those not attending live.

The TA and instructor also hold weekly office hours, intended as a time for you to get one-on-one help with WeBWorK and ask other course-related questions (with the instructor). Office hours are at the following times:

- Prof. Dummit: 3:15pm-4:30pm MR, or by appointment
- TA: 3:30pm-4:30pm T.

Office hours are held via Zoom and are not recorded.

Miscellaneous Info

Here is some other miscellaneous information:

- The instructor will write lecture notes for the course (in lieu of an official textbook) as the semester progresses. The course will roughly follow the presentation in Larsen and Marx's "An Introduction to Mathematical Statistics and its Applications" (5th edition), but it is not necessary to purchase the textbook.
- Course prerequisites: Math 1242, 1252, or 1342 (Calculus 2).
- Collaboration and technology: You are free to use calculators and computer technology for homework problems, and calculators are allowed on exams. You are allowed to work on, and discuss, homework assignments together, as long as the actual submissions are your own work. Collaboration is, of course, not allowed on exams.

Other Boilerplate, I

- Statement on Academic Integrity: A commitment to the principles of academic integrity is essential to the mission of Northeastern University. Academic dishonesty violates the most fundamental values of an intellectual community and undermines the achievements of the entire University. Violations of academic integrity include (but are not limited to) cheating on assignments or exams, fabrication or misrepresentation of data or other work, plagiarism, unauthorized collaboration, and facilitation of others' dishonesty. Possible sanctions include (but are not limited to) warnings, grade penalties, course failure, suspension, and expulsion.

Other Boilerplate, II

- Statement on Accommodations: Any student with a disability is encouraged to meet with or otherwise contact the instructor during the first week of classes to discuss accommodations. The student must bring a current Memorandum of Accommodations from the Office of Student Disability Services.
- Statement on Classroom Behavior: Disruptive classroom behavior will not be tolerated. In general, any behavior that impedes the ability of your fellow students to learn will be viewed as disruptive.
- Statement on Inclusivity: Faculty are encouraged to address students by their preferred name and gender pronoun. If you would like to be addressed using a specific name or pronoun, please let your instructor know.

Other Boilerplate, III

- Statement on Evaluations: Students are requested to complete the TRACE evaluations at the end of the course.
- Miscellaneous Disclaimer: The instructor reserves the right to change course policies, including the evaluation scheme of the course (e.g., in the event of natural disaster or global pandemic).[1] Notice will be given in the event of any substantial changes.

[1] This is a verbatim quote from my syllabus template from 10 years ago, just in case you're wondering.

Awkward Transition To Doing Actual Math Now

Pause here for questions about course logistics.

Note to self: don't read this slide out loud.

Overview of §1: Counting and Probability

Our first topic in Probability and Statistics is to discuss probability, which quantifies how likely it is that a particular event will occur.

- We begin with a brief review of various basic properties of sets and set operations (this lecture).
- Then we introduce basic counting principles (also this lecture), permutations, combinations, and binomial coefficients (next lecture), which are all ultimately grounded in properties of sets.
- With those basics in hand, we can develop the fundamentals of discrete probability (after that), which in turn rely heavily on counting principles and set properties.

Let's Talk About Sets, I

"Definition"

A set is a well-defined collection of distinct elements.

- The elements of a set can be essentially anything: integers, real numbers, other sets, people.
- Sets are generally denoted by capital or script letters, and when listing the elements of a set, curly brackets $\{\cdot\}$ are used.
- Sets do not have to have any elements: the empty set $\emptyset = \{\ \}$ is the set with no elements at all.
- Two sets are the same precisely if all of their elements are the same. The elements in a set are not ordered, and no element can appear in a set more than once: thus $\{1, 4, 3\}$ and $\{3, 1, 4\}$ are the same set.

Let's Talk About Sets, II

There are two primary ways to describe a set.

Method 1 is to list all the elements (implicitly or explicitly).
Examples:

- $A = \{1, 2, 4, 5\}$ is the set containing the four numbers 1, 2, 4, and 5.
- $B = \{1, 2, 3, 4, \ldots, 100\}$ is the set containing the first 100 positive integers.

Method 2 is to describe properties of the elements.
Examples:

- The S of solutions x to the equation $x^2 = 1$ has two elements: $S = \{1, -1\}$.
- The set T of TAs for this course has one element $T = \{\text{Jiewei Feng}\}$.

Let's Talk About Sets, III

The most basic query we can make of a set is whether a particular object x is an element or not:

Definition (Notation)

If S is a set, $x \in S$ means "x is an element of S", and $x \notin S$ means "x is not an element of S".

Examples:

- For $S = \{1, 2, 5\}$ we have $1 \in S$ and $5 \in S$ but $3 \notin S$ and $\pi \notin S$.
- If S is the set of English words starting with the letter A, we have apple $\in S$ and antlers $\in S$ and anaphylaxis $\in S$, while potatoes $\notin S$ and $7 \notin S$ and $1000 \notin S$.

Let's Talk About Sets, IV

The situation where one set contains all the elements of another set is important:

Definition

If A and B are two sets with the property that every element of A is also an element of B, we say A is a <u>*subset*</u> *of B (or that A* <u>*is contained in*</u> *B) and write $A \subseteq B$.*

<u>Examples</u>:

- If $A = \{1, 2, 3\}$, $B = \{1, 4, 5\}$, and $C = \{1, 2, 3, 4, 5\}$, then $A \subseteq C$ and $B \subseteq C$ but neither A nor B is a subset of the other.
- If S is the set of all English words and T is the set of all English words starting with the letter t, then $T \subseteq S$.
- If A is any set, then the empty set $\emptyset$ is contained in A.

Let's Talk About Sets, V

The number of elements in a set is also quite important:

Definition

If A is any set, the <u>cardinality</u> of A, denoted $\#A$ or $|A|$, is the number of distinct elements in A.

<u>Examples</u>:

- For $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8, 10, \dots, 100\}$, then $\#A = 3$ and $\#B = 50$.
- The cardinality of the empty set $\emptyset$ is 0.
- The cardinality of the set $\{1, 2, 3, 4, \dots\}$ of positive integers is ∞.

It is easy to see that if $A \subseteq B$, then $\#A \leq \#B$.

Set Operations, I: Intersection and Union

Given two sets, we now consider the elements common to both sets, and also the elements in at least one of the two sets:

Definition

If A and B are two sets, then the <u>intersection</u> $A \cap B$ is the set of all elements contained in both A and B. The <u>union</u> $A \cup B$ is the set of all elements contained in either A or B (or both).

<u>Examples</u>:

- If $A = \{1, 2, 3\}$ and $B = \{1, 4, 5\}$, then $A \cap B = \{1\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$.
- If $C = \{0, 2, 6\}$ and $D = \{3, 5\}$, then $C \cap D = \emptyset$ and $C \cup D = \{0, 2, 3, 5, 6\}$.

It is easy to see that if $A \subseteq B$, then $A \cap B = A$ and $A \cup B = B$.

Set Operations, II: Intersection and Union

Examples (continued):

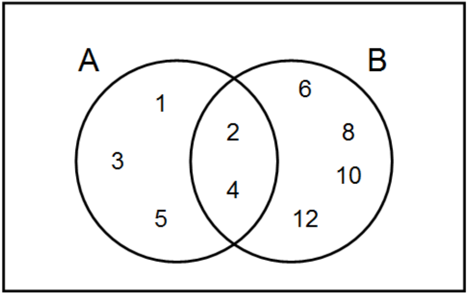
- If $E = \{2, 4, 6, 8, \ldots\}$ is the set of all positive even integers and $S = \{1, 4, 9, 16, \ldots\}$ is the set of all positive perfect squares, then $E \cap S = \{4, 16, 36, 64, \ldots\}$ is the set of all even perfect squares.
- If $A = \{1, 2, 3\}$, $B = \{1, 3, 4\}$, and $C = \{1, 3, 9\}$, then $A \cap B \cap C = \{1, 3\}$ while $A \cup B \cup C = \{1, 2, 3, 4, 9\}$.
- If $R = \{1, 2, 3\}$, $S = \{1, 3, 4\}$, and $T = \{1, 3, 9\}$, then $(R \cap S) \cup T = \{1, 3\} \cup \{1, 3, 9\} = \{1, 3, 9\}$ while $R \cap (S \cup T) = \{1, 2, 3\} \cap \{1, 3, 4, 9\} = \{1, 3\}$.

The last example shows that we cannot mix unions and intersections without specifying the order of operations.

Set Operations, III: Venn Diagrams

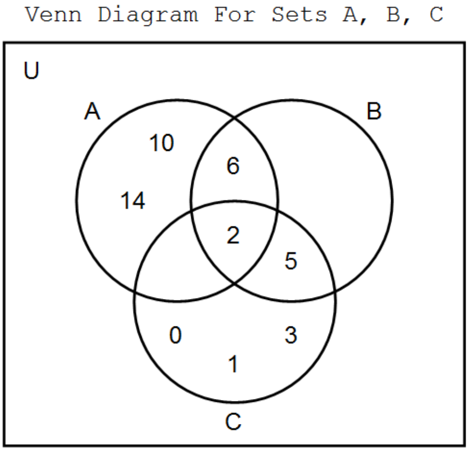
Venn diagrams are very useful for visualizing unions and intersections. Here is a Venn diagram for the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10, 12\}$:

Venn Diagram For Sets A and B



Set Operations, IV: Venn Diagrams

Here is a Venn diagram for the sets $A = \{2, 6, 10, 14\}$, $B = \{2, 5, 6\}$, $C = \{0, 1, 2, 3, 5\}$:



Set Operations, V: Venn Diagrams

There is a simple relation between the sizes of two sets and the sizes of their union and intersection:

Theorem (Sizes of Unions and Intersections)

If A and B are any sets, then $\#(A \cup B) + \#(A \cap B) = \#A + \#B$.

Set Operations, V: Venn Diagrams

There is a simple relation between the sizes of two sets and the sizes of their union and intersection:

Theorem (Sizes of Unions and Intersections)

If A and B are any sets, then $\#(A \cup B) + \#(A \cap B) = \#A + \#B$.

Proof: Both sides count every element in $A \cap B$ twice and every other element once, so they are equal. (Try drawing a Venn diagram if you can't see why this is true.)

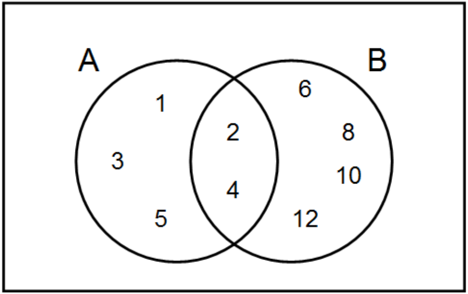
Set Operations, VI: Venn Diagrams

Example: For $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10, 12\}$, verify that $\#(A \cup B) + \#(A \cap B) = \#A + \#B$.

Set Operations, VI: Venn Diagrams

Example: For $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10, 12\}$, verify that $\#(A \cup B) + \#(A \cap B) = \#A + \#B$.

Venn Diagram For Sets A and B



Using the Venn diagram we can see $\#A = 5$, $\#B = 6$, $\#A \cap B = 2$, $\#A \cup B = 9$, and indeed $9 + 2 = 5 + 6$.

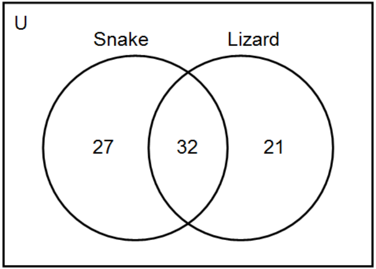
Set Operations, VII: Venn Diagrams

<u>Example</u>: A survey of 80 reptile owners shows that 59 own a snake and 53 own a lizard, and none have any other pets. How many owners have both a snake and a lizard?

Set Operations, VII: Venn Diagrams

Example: A survey of 80 reptile owners shows that 59 own a snake and 53 own a lizard, and none have any other pets. How many owners have both a snake and a lizard?

- Let S be the set of snake owners and L be the set of lizard owners. Then $\#S = 59$, $\#L = 53$, and $\#(S \cup L) = 80$.
- Thus $\#(S \cap L) = \#S + \#L - \#(S \cup L) = 59 + 53 - 80 = 32$, meaning that 32 owners have both a snake and a lizard.



Set Operations, VIII: Universal Sets

In many contexts, it is useful to think of all the sets we are discussing as being subsets of some particular larger set S, which we refer to as a universal set of elements under consideration.

- In general, we must always specify precisely what this universal set U is, unless it is clear from context. For example, if we are discussing sets of integers, a sensible choice is to take U to be the set of integers, but there is no reason we couldn't instead take U to be the set of all real numbers.
- It might seem to be convenient to use the same universal set in all contexts, but it turns out that assuming the existence of a general "universal set" of all possible elements leads to logical contradictions[2].

[2] If you want to learn more about this, look up "Russell's Paradox", and/or take Math 1365!

Set Operations, IX: The Rise of ~~Skywalker~~ Complements

If we have chosen a suitable universal set U and A is a subset of U, then we may speak of the elements of U not in A.

Definition

If U is a universal set and $A \subseteq U$, then the complement of A (as a subset of U), denoted as A^c, is the set of elements of U not in A.

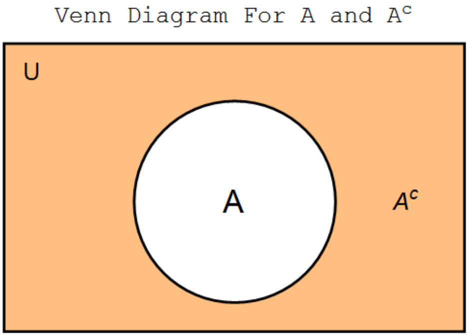
Example:

- With universal set $U = \{1, 2, 3, 4, 5, 6\}$, if $A = \{1, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$, then $A^c = \{2, 5, 6\}$ and $B^c = \emptyset$.

Other notations sometimes used for the complement of A as a subset of U include A', $\overline{A}$, $U \backslash A$, and $U - A$.

Set Operations, X: Complements

In a Venn diagram, we always identify the universal set U in the diagram (usually by marking it in the corner), and represent A^c as the area outside the region marked as A:



Set Operations, XI: Complements

We have a few basic properties of complements that follow from the definition:

- Observe that for any set A, we have $A \cup A^c = U$, because every element is either in A or not in A.
- Also, $A \cap A^c = \emptyset$ because by definition no element is both in A and not in A.
- Using the two properties above and the union-intersection cardinality formula, we see that $\#A + \#A^c = \#(A \cup A^c) + \#(A \cap A^c) = \#U$.
- This means, if A is finite, then $\#A^c = \#U - \#A$.

It is also true that $(A^c)^c = A$ (try working this one out yourself!).

Set Operations, XII: Example

Example: In a literature class, a total of 45 short stories are read:

- 25 are romantic, 18 are science fiction, and 14 are dystopian.
- Furthermore, 8 of the science fiction stories are romantic, 2 of which are also dystopian.
- There are 7 dystopian science fiction stories.
- Every dystopian story is either romantic or science fiction.

Determine the number of short stories that are (i) romantic or dystopian, (ii) non-dystopian science fiction, and (iii) none of the three categories.

Set Operations, XII: Example

<u>Example</u>: In a literature class, a total of 45 short stories are read:

- 25 are romantic, 18 are science fiction, and 14 are dystopian.
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- There are 7 dystopian science fiction stories.
- Every dystopian story is either romantic or science fiction.

Determine the number of short stories that are (i) romantic or dystopian, (ii) non-dystopian science fiction, and (iii) none of the three categories.

- We can solve this kind of problem using a Venn diagram.

Set Operations, XII: Example

Example: In a literature class, a total of 45 short stories are read:

- 25 are romantic, 18 are science fiction, and 14 are dystopian.
- Furthermore, 8 of the science fiction stories are romantic, 2 of which are also dystopian.
- There are 7 dystopian science fiction stories.
- Every dystopian story is either romantic or science fiction.

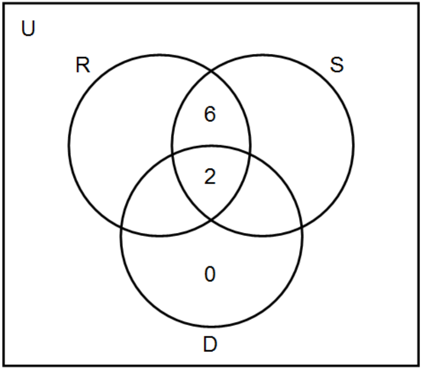
Set Operations, XII: Example

Let U be all 45 short stories, R the romantic stories, S the science-fiction stories, and D the dystopian stories. Then

- There are 45 stories, so $\#U = 45$.
- There are 25 romantic stories, so $\#R = 25$.
- There are 18 science-fiction stories, so $\#S = 18$.
- There are 14 dystopian stories, so $\#D = 14$.
- There are 8 romantic science-fiction stories, so $\#(R \cap S) = 8$. Of these 2 are dystopian, so $\#(R \cap S \cap D) = 2$, hence also $\#(R \cap S \cap D^c) = 6$,
- There are 7 dystopian science-fiction stories, so $\#(S \cap D) = 7$.
- Finally, $\#(D \cap R^c \cap S^c) = 0$ because there are no stories that are dystopian, but not romantic and not science fiction.

Set Operations, XIII: Example

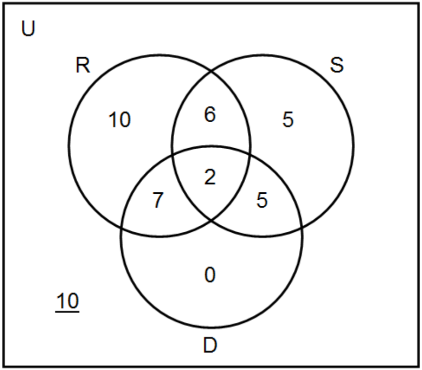
Now we can assemble this information into a Venn diagram:



Next, we can use the unused information to fill in the other entries.

Set Operations, XIV: Example

The end result is shown below:



Now we simply read off the desired answers: $\#(R \cup D) = 30$, $\#(S \cap D^c) = 11$, $\#(R^c \cap S^c \cap D^c) = 10$.

Set Operations, XV: Cartesian Products

There is one more important set operation that we will need to use:

Definition

If A and B are any sets, the Cartesian product is the set $A \times B$ consisting of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Examples:

- If $A = \{1, 2\}$ and $B = \{1, 3, 5\}$, then $A \times B = \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5)\}$.
- If $A = \{H, T\}$ then $A \times A = \{(H, H), (H, T), (T, H), (T, T)\}$.

As is easy to guess based on the examples, there is a simple formula for the cardinality of a Cartesian product: for any sets A and B we have $\#(A \times B) = \#A \cdot \#B$.

Set Operations, XVI: Cartesian Products

A common use of the Cartesian product is to list all possible outcomes when one event is followed by another.

- The second example on the previous slide indicates the possible outcomes of flipping one coin followed by flipping another coin.
- Another possibility would be to list all the results of rolling a standard six-sided die twice in a row.
- We can also take Cartesian products of more than two sets, which simply is the set of all the appropriate ordered tuples.
- For example, the Cartesian product $A \times B \times C$ is the set of all ordered triples (a, b, c) where $a \in A$, $b \in B$, and $c \in C$.

Our goal is now to use some of these properties of sets to solve counting problems, since these techniques will be very useful when we start working with probability.

Counting Basics: Addition Principle

Our first basic counting principle is called the "addition principle":

Principle (Addition Principle)

When choosing among n disjoint options labeled 1 through n, if option i has a_i possible outcomes for each $1 \leq i \leq n$, then the total number of possible outcomes is $a_1 + a_2 + \cdots + a_n$.

Example:

- If a restaurant offers 5 main courses with chicken, 6 main courses with beef, and 12 vegetarian main courses, then the total possible number of main courses is $5 + 6 + 12 = 23$.

The addition principle can be justified using our results about sets: if A_i corresponds to the set of outcomes of option i, then because all of the different options are disjoint,
$\#(A_1 \cup A_2 \cup \cdots \cup A_n) = \#A_1 + \#A_2 + \cdots + \#A_n$.

Counting Basics: Multiplication Principle

Our other basic counting principle is the "multiplication principle":

Principle (Multiplication Principle)

When making a sequence of n independent choices, if step i has b_i possible outcomes for each $1 \leq i \leq n$, then the total number of possible collections of choices is $b_1 \cdot b_2 \cdot \cdots \cdot b_n$.

<u>Examples</u>:

- If a fair coin is tossed (2 possible outcomes) and then a fair 6-sided die is rolled (6 possible outcomes), the total number of possible results is $2 \cdot 6 = 12$.
- If a fair coin is tossed 4 times, the total number of possible results is $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

The multiplication principle can also be justified using our results about sets: if B_i corresponds to the set of outcomes of choice i, then $\#(B_1 \times B_2 \times \cdots \times B_n) = \#B_1 \cdot \#B_2 \cdot \cdots \cdot \#B_n$.

Counting Basics: Examples, I

<u>Example</u>: Determine the number of possible outcomes from rolling a 6-sided die 5 times in a row.

Counting Basics: Examples, I

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- Each individual roll has 6 possible outcomes. Thus, by the multiplication principle, the number of possible sequences of 5 rolls is $6^5 = 7776$.

Example: Determine the number of possible ways to write a sequence of four digits 0-9 in a row.

Counting Basics: Examples, I

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Example: Determine the number of possible ways to write a sequence of four digits 0-9 in a row.

- Each individual digit has 10 possible choices. Thus, by the multiplication principle, the number of possible sequences of 4 digits is $10^4 = 10000$.

Counting Basics: Examples, II

Example: An ice creamery offers 25 different flavors. Each order of ice cream may be served in either a sugar cone, a waffle cone, or a dish, and may have 2 or 3 scoops (which must be the same flavor). Also, any order may come with a cherry or nuts (or neither), but not both. How many different orders are possible?

Counting Basics: Examples, II

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- We tabulate all of the possible choices separately.
- First, we choose an ice cream flavor: 25 options.
- Then we choose a sugar cone, waffle cone, or dish: 3 options.
- Next we choose the number of scoops: 2 options.
- Finally, we choose either a cherry, nuts, or neither: 3 options.
- By the multiplication principle, the total number of possible orders is $25 \cdot 3 \cdot 2 \cdot 3 = 450$.

Counting Basics: Examples, III

Example: An artist is painting the four walls of her kitchen. The north and south walls can each be eggshell, cream, or snow, while the east and west walls can each be blue, cyan, turquoise, or emerald. How many ways can the kitchen be painted if any combination of wall colors is acceptable?

Counting Basics: Examples, III

Example: An artist is painting the four walls of her kitchen. The north and south walls can each be eggshell, cream, or snow, while the east and west walls can each be blue, cyan, turquoise, or emerald. How many ways can the kitchen be painted if any combination of wall colors is acceptable?

- We tabulate all of the possible choices separately.
- The north wall has 3 color options.
- The south wall has 3 color options.
- The east wall has 4 color options.
- The west wall has 4 color options.
- By the multiplication principle, the total number of possible paint combinations is $3 \cdot 3 \cdot 4 \cdot 4 = 144$.

Counting Basics: Examples, IV

Example: Determine the number of subsets of the set $\{1, 2, \ldots, n\}$.

Counting Basics: Examples, IV

Example: Determine the number of subsets of the set $\{1, 2, \ldots, n\}$.

- Here, we can completely characterize a subset of $\{1, 2, \ldots, n\}$ by listing, for each $k \in \{1, 2, \ldots, n\}$, whether $k \in S$ or $k \notin S$.
- This means that for each of the n elements in the original set, we make a selection among 2 different options.
- So, by the multiplication principle, the number of possible ways of making this sequence of n choices is $\boxed{2^n}$.

Counting Basics: Examples, V

Example: At a car dealership, Brand X sells 11 different models of cars each of which comes in 20 different colors, while Brand Y sells 6 different models of cars each of which comes in 5 different colors. How many different possible car options (including brand, model, and color) can be purchased at the dealership?

Counting Basics: Examples, V

Example: At a car dealership, Brand X sells 11 different models of cars each of which comes in 20 different colors, while Brand Y sells 6 different models of cars each of which comes in 5 different colors. How many different possible car options (including brand, model, and color) can be purchased at the dealership?

- If a Brand X car is purchased, there are 11 choices for the model and 20 choices for the color, so by the multiplication principle there are $11 \cdot 20 = 220$ possible options in this case.
- If a Brand Y car is purchased, there are 6 choices for the model and 5 choices for the color, so by the multiplication principle there are $6 \cdot 5 = 30$ possible options in this case.
- Since these two cases are disjoint, in total there are $220 + 30 = 250$ possible car options.

Counting Basics: Examples, VI

Example: A local United States telephone number has 7 digits and cannot start with 0, 1, or the three digits 555. How many such telephone numbers are possible?

Counting Basics: Examples, VI

Example: A local United States telephone number has 7 digits and cannot start with 0, 1, or the three digits 555. How many such telephone numbers are possible?

- The first digit has 8 possibilities (namely, the digits 2 through 9 inclusive) and the other six digits each have 10 possibilities. Thus, by the multiplication principle, there are $8 \cdot 10^6 = 8\,000\,000$ total telephone numbers.
- However, we have included the numbers starting with 555: each of these has 10 choices for each of the last 4 digits, for a total of $10^4 = 10\,000$ telephone numbers.
- Subtracting the disallowed numbers yields a total of $8\,000\,000 - 10\,000 = 7\,990\,000$ local telephone numbers.

Counting Basics: Examples, VII

Example (again): A local United States telephone number has 7 digits and cannot start with 0, 1, or the three digits 555. How many such telephone numbers are possible?

Counting Basics: Examples, VII

Example (again): A local United States telephone number has 7 digits and cannot start with 0, 1, or the three digits 555. How many such telephone numbers are possible?

- Each of the 7 digits has 10 possibilities, which would give a total of $10^7 = 10\,000\,000$ telephone numbers.
- However, we must subtract the disallowed ones starting with 0, 1, or 555.
- The numbers starting with 0 have 1 choice for the first digit and 10 choices for the other 6, a total of $1 \cdot 10^6 = 1\,000\,000$.
- Similarly, there are also $1\,000\,000$ numbers starting with 1.
- The numbers starting with 555 have 1 choice for each of the first 3 digits and 10 for the last 4, a total of $1^3 \cdot 10^4 = 10\,000$.
- Excluding the disallowed numbers yields $10\,000\,000 - 2 \cdot 1\,000\,000 - 10\,000 = 7\,990\,000$ possibilities.

Summary

We discussed the logistics for Math 3081.

We discussed sets and basic set operations (subsets, cardinality, union, intersection, complement, and Cartesian product) and how to visualize them with Venn diagrams.

We introduced basic counting principles.

Next lecture: Permutations, combinations, events, sample spaces.