Math 3081, Probability and Statistics, Summer 2, 2019

Test 2

Solutions

Please **TYPE** your **First Name** and **Last Name**:

Please show all your work, justify your conclusions and give the final answer as a number, either fraction or decimal.

Remember:

- diagram, tree or some other picture is not a solution, but it can help you;
- the solution **must** include the formula(s) that you are using, not just the numbers and operations between them;
- please write down your **final answer**.

Problem 1. (20 pts)
Given the joint PDF of discrete random variables X and Y find the requested parameters.

$X\backslash Y$	0	2	3	$p_X(x)$	$xp_X(x)$
0	0.5	0.1	0	0.6	0
1	0.2	0.1	0.1	0.4	0.4
$p_Y(y)$	0.7	0.2	0.1		
$yp_Y(y)$	0	0.4	0.3		

(a) (5 pts) Find marginal PDFs of X and Y (Hint: you can fill them in the table above);

Please look at the table

(b) (10 pts) Compute expected values of X and Y (Hint: you can use the table above);

$EX = 0 + 0.4 = 0.4$

$EY = 0 + 0.4 + 0.3 = 0.7$

(c) (5 pts) Find the covariance between X and Y.

$Cov(X,Y) = E(XY) - EX \cdot EY = 0 \cdot 0 \cdot 0.5 + 0 \cdot 2 \cdot 0.1 + 0 \cdot 3 \cdot 0 +$

$+ 1 \cdot 0 \cdot 0.2 + 1 \cdot 2 \cdot 0.1 + 1 \cdot 3 \cdot 0.1 - 0.4 \cdot 0.7 = 0.5 - 0.28 = 0.22$

Problem 2. (10 pts)

A box contains 3 number cards with 1, 2, 3 on them. You draw one card *without replacement*, then the second one is drawn. Let X be the minimal number on both cards, and $Y-$ the maximal number. Write down the joint PDF for X and Y as a table. (Hint: determine your sample space and find possible values of X and Y.)

Sample space	Probability	X	Y
(1,2)	1/6	1	2
(2, 1)	1/6	1	2
(3, 1)	1/6	1	3
(1, 3)	1/6	1	3
(2, 3)	1/6	2	3
(3, 2)	1/6	2	3

X \ Y	2	3
1	$\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$	$\frac{1}{3}$
2	0	$\frac{1}{3}$

Problem 3. (25 pts)

Suppose the PDF of the continuous random variable X is given by the formula $f_X(x) = c(x+1)$, $x \in [0,1]$, for some unknown parameter c.

(a) (5 pts) Find the value of the constant c. (Remember the definition of PDF?)

$$1=\int_{-\infty}^{+\infty} f_X(x)\,dx = c\int_0^1 (x+1)\,dx = c\cdot\left(\frac{x^2}{2}+x\right)\Big|_0^1 = \frac{3}{2}c \Rightarrow c = \frac{2}{3}$$

(b) (5 pts) Find the CDF $F_X(x)$ for X. (Hint: consider different cases of x.)

$$F_X(x) = \begin{cases} 0, & x<0 \\ \frac{2}{3}\int_0^x (t+1)\,dt, & 0\le x\le 1 \\ 1, & x>1 \end{cases} = \begin{cases} 0, & x<0 \\ \frac{x^2}{3}+\frac{2x}{3}, & x\in[0,1] \\ 1, & x>1 \end{cases}$$

(c) (10 pts) Compute the probability that $X \in [-0.5, 0.5]$. (Use either PDF or CDF)

$$P(-0.5\le X\le 0.5) = F_X(0.5) - F_X(-0.5) = \left(\frac{1}{12}+\frac{1}{3}\right) - 0 = \frac{5}{12}$$

≈ 0.4167

(d) (5 pts) Find the expected value and variance of X.

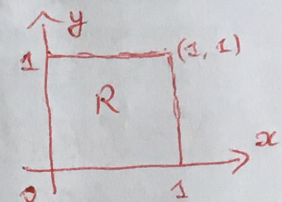
$$EX = \frac{2}{3}\int_0^1 x\cdot(x+1)\,dx = \frac{2}{3}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\Big|_0^1 = \frac{5}{9} \approx 0.5556$$

$$E(X^2) = \frac{2}{3}\int_0^1 x^2\cdot(x+1)\,dx = \frac{2}{3}\left(\frac{x^4}{4}+\frac{x^3}{3}\right)\Big|_0^1 = \frac{7}{18} \approx 0.3889$$

$$\mathrm{Var}\,X = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162} \approx 0.08025$$

Problem 4. (15 pts)

Suppose the joint PDF of the continuous random variables X, Y is given by the formula $f_{X,Y}(x, y) = c(x + y)$, $x, y \in [0, 1]$, for some unknown parameter c.



(a) (5 pts) Find the value of the constant c.

$$1 = \iint_R f_{X,Y}(x,y)\,dx\,dy = c\int_0^1 \left(\int_0^1 (x+y)\,dy \right) dx =$$

$$= c\int_0^1 \left(xy + \frac{y^2}{2} \right)\Big|_{y=0}^{y=1} dx = c\int_0^1 \left(x + \frac{1}{2}\right) dx = c\left(\frac{x^2}{2} + \frac{x}{2}\right)\Big|_0^1 = c \Rightarrow \boxed{c = 1}$$

(b) (5 pts) Find the marginal pdfs;

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)\Big|_0^1 = x + \frac{1}{2}, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^1 (x+y)\,dx = \left(\frac{x^2}{2} + xy\right)\Big|_0^1 = y + \frac{1}{2}, \quad 0 \le y \le 1$$

(c) (5 pts) Compute the probability that $X \in [0, 1/2]$.

$$P\left(0 \le X \le \frac{1}{2}\right) = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} \left(x + \frac{1}{2}\right) dx = \frac{x^2 + x}{2}\Big|_0^{1/2} = \frac{3}{8}$$

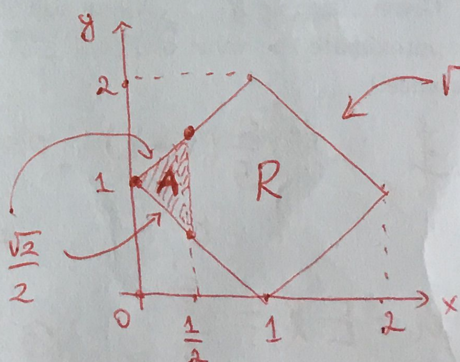
Problem 5. (5 pts)

Let (X, Y) be the coordinate of a random point picked from the square with vertices (1,0),(0,1),(1,2),(2,1). Find $P(X \le 1/2)$. (Hint: it is geometric probability.)

R - square

A - triangle

$$P\left(X \le \frac{1}{2}\right) = \frac{\text{Area } A}{\text{Area } R} = \frac{\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}}{(\sqrt{2})^2} = \frac{1}{8}$$



Problem 6. (5 pts)

Let X be a normal variable with a mean $\mu = 212$ and a standard deviation $\sigma = 44.6$. If you take a random sample of size 80, find $P(\overline{X} < 220)$, where $\overline{X}$ is a sample mean. (Hint: what are the mean and variance of $\overline{X}$?)

$$\left. \begin{array}{l} X \sim N(212, 44.6^2) \\ n = 80 \end{array} \right| \Rightarrow \overline{X} \sim N\left(212, \frac{44.6^2}{80}\right)$$

$$P(\overline{X} < 220) = \text{normalcdf}\left(-10000, 220, 212, \sqrt{\frac{44.6^2}{80}}\right) \approx$$

something very-very negative

≈ 0.9457

Problem 7. (10 pts)
38% of people in the country have type O+ blood. If you randomly chose 400 people, use the *normal approximation with continuity correction* to estimate (where X is the number with type O+): $P(160 \le X \le 200)$. (Hint: what distribution has X originally?)

$X \sim \text{Binomial}(n, p) = \text{Binomial}(400, 0.38)$

$n = 400,\ p = 0.38 \Rightarrow q = 1 - p = 0.62$

CLT: $X \sim N(np, npq) = N(152, 94.24)$

$P(160 \le X \le 200) \underset{\text{corect.}}{\overset{\text{cont.}}{=}} \text{normalcdf}(160 - 0.5, 200 + 0.5, 152, \sqrt{94.24}) =$

$= \boxed{0.2199}$

Problem 8. (5 pts)
A store gets 15 customers per hour on average. If the number of customers is Poisson, find the probability that the store gets five in the next 12 minutes.

15 per hour $\Rightarrow \frac{15}{5} = 3$ per 12 minutes.

$\lambda = 3$

$P_X(5) = \frac{\lambda^5 e^{-\lambda}}{5!} = \frac{81}{40e^3} \approx \boxed{0.1008}$

Problem 9. (5 pts)
Given a sample $X_1, \ldots, X_{30}$ of size 30 from the continuous distribution with PDF given in Problem 3 approximate the value of $P(1/4 \le \overline{X} \le 3/4)$. (Hint: you already should have expected value and variance!)

$$f_X(x) = \begin{cases} \frac{2}{3}(x+1), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$\mu = EX = \frac{5}{9}, \quad \sigma^2 = Var(X) = \frac{13}{162}, \quad n = 30$

$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(\frac{5}{9}, \frac{13}{4860}\right) = N(0.5556, 0.0027)$

$P\left(\frac{1}{4} \le \overline{X} \le \frac{3}{4}\right) = \text{normalcdf}(0.25, 0.75, 0.5556, 0.052) \approx$

$\approx \boxed{0.999907}$