Math 3081, Probability and Statistics, Summer 2, 2019

Test 1

Solutions

Please **TYPE** your **First Name** and **Last Name**:

Please show all your work, justify your conclusions and give the final answer as a number, either fraction or decimal.

Remember:

– diagram, tree or some other picture is not a solution, but it can help you;

– the solution **must** include the formula(s) that you are using, not just the numbers and operations between them;

– please write down your **final answer**.

Problem 1. (5 pts)

Let A and B be any two events defined on S. Suppose that $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.1$. What is the probability that A or B but not both occur? (Remark: please justify your answer by formulas, not just by Venn diagram!)

$$P((A \setminus B) \cup (B \setminus A)) = P(A \cup B) - P(A \cap B) =$$
$$= P(A) + P(B) - 2P(A \cap B) =$$
$$= 0.4 + 0.5 - 2 \cdot 0.1 = \underline{\underline{0.7}}$$

Problem 2.

Let $P(A \cap B) = 0.2$, $P(A|B) = 0.5$, and $P(B|A) = 0.4$. Justify your answers to the following questions:

(a) (6 pts) find $P(A)$ and $P(B)$;

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(B) = \frac{P(A \cap B)}{P(A|B)} = \frac{0.2}{0.5} = \underline{\underline{0.4}}$$
$$P(A) = \frac{P(A \cap B)}{P(B|A)} = \frac{0.2}{0.4} = \underline{\underline{0.5}}$$

(b) (2 pts) are A and B disjoint?

$\underline{\underline{\text{No}}}$, because if $A \cap B = \emptyset$ then $P(A \cap B) = 0$ $- ?!$

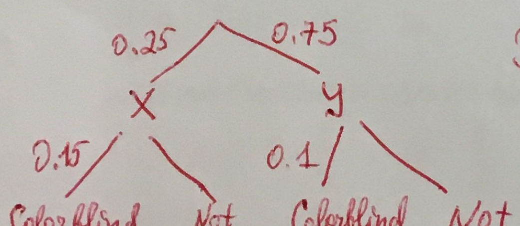
(c) (2 pts) are A and B independent?

$$\left.\begin{array}{l} P(A \cap B) = 0.2 \\ P(A)\,P(B) = 0.5 \cdot 0.4 = 0.2 \end{array}\right| \Rightarrow \underline{\underline{\text{Yes}}}!$$

Problem 3.

In the population X it is known that 15% of animals are colorblind, and 10% are colorblind in the polulation Y. Assume that the probability that the randomly picked animal is from the population X equals 25%, and 75% that it is from the population Y.

(a) (8 pts) What is a probability that a randomly selected animal is colorblind? (Hint: draw a tree.)



$$P(\text{Colorblind}) = 0.15 \cdot 0.25 + 0.1 \cdot 0.75 =$$
$$= \underline{\underline{0.1125}} \quad \left(\text{or } \underline{\underline{\frac{9}{80}}}\right)$$

(b) (7 pts) If a randomly selected animal is colorblind, what is the probability that this animal is a from the population X? (Hint: use Bayes' theorem.)

$$P(X \mid \text{Colorblind}) = \frac{P(\text{Colorblind} \mid X)\,P(X)}{P(\text{Colorblind})} = \frac{0.15 \cdot 0.25}{0.1125} = \frac{1}{3}$$

Problem 4.

Suppose that the fair dice is tossed twice. Let i be a result of the first toss, and j – of the second toss.

(a) (5 pts) What is the probability that $i + j = 8$?

26, 35, 44, 53, 62

$P(i+j=8) = 5/36 \approx 0.13889$

(b) (5 pts) What is the probability that $i + j = 10$ given that $j \geq 5$? (Hint: Write down all pairs that satisfy your events.)

$P(i+j=10 \mid j \geq 5) = \frac{P(i+j=10 \text{ and } j \geq 5)}{P(j \geq 5)} =$

$= \frac{P\{(5,5),(4,6)\}}{P\{*5 \mid * - \text{any}\}} = \frac{2/36}{12/36} = \frac{1}{6} = 0.1667$

Problem 5. (5 pts)

A fair coin is tossed three times, and k is the number of heads that occured. What is the probability that $k \geq 2$ if it is known that $k \leq 2$ (i.e. at most two heads have occurred)? (Hint: use conditional probability.)

A – event that $k \leq 2$

B – event that $k \geq 2$

$S = \begin{Bmatrix} HHH & THH \\ HHT & THT \\ HTH & TTH \\ HTT & TTT \end{Bmatrix}$

$A = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$B = \{HHH, HHT, HTH, THH\}$

$A \cap B = \{HHT, HTH, THH\}$

$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{3/8}{7/8} = 3/7$

(or 0.4286)

Problem 6.

Four men and four women are to be seated in a row of chairs numbered 1 through 8.

(a) (5 pts) In how many ways these 8 people can sit in a row?

$40320 = 8!$ ways

(b) (10 pts) In how many ways these 8 people can sit if the men are required to sit in alternate chairs? (Hint: consider both situations MWMWMWMW and WMWMWMWM)

Case 1: MWMWMWMW

$4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 4! \cdot 4!$ combinations

Case 2: WMWMWMWM – $4! \cdot 4!$ combinations

Together $2 \cdot 4! \, 4! = \boxed{1152}$

Problem 7. (10 pts)
Five cards are dealt from a standard 52-card deck. What is the probability that the sum of the faces on the five cards is 48 or more? Assume that Jack, Queen, King and Ace are not representing any number. (Hint: what combinations of 5 cards can make a sum 48 or more?)

Possible values : 48, 49

(50 cannot be realized using 5 cards)

48 — 10, 10, 10, 10, 8 — $C_4^4 C_4^1 = 4$

48 — 10, 10, 10, 9, 9 — $C_4^3 C_4^2 = 24$

49 — 10, 10, 10, 10, 9 — $C_4^4 C_4^1 = 4$

$P = \frac{4+4+24}{C_{52}^5} = \frac{32}{C_{52}^5}$ (or 0.0000123)

Problem 8.
A fair die is rolled two times. Let the random variable X denote the number of even dices that appeared.

(a) (5 pts) Find the range $Ran(X)$ of X;

$Ran(X) = \{0, 1, 2\}$

0 — (odd ; odd) — 9 ways

1 — (odd, even), (even, odd) — 18 ways

2 — (even, even) — 9 ways

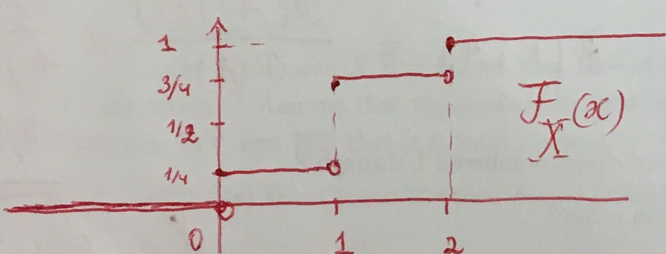
(b) (10 pts) Find the PDF $p_X(x)$ of X; (write PDF as a table)

X	0	1	2
p_X	1/4	1/2	1/4

$P\{X=0\} = \frac{9}{36} = \frac{1}{4} = P\{X=2\}$

$P\{X=1\} = \frac{18}{36} = \frac{1}{2}$

(c) (10 pts) Find and graph the plot of the CDF $F_X(x)$ of X; (Remember that CDF need to be defined for **all real numbers**)



(d) (3 pts) Calculate the expected value $E(X)$ of X;

$EX = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

(e) (2 pts) Find the variance $Var(X)$ of X.

$EX^2 = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} = \frac{3}{2}$

$Var\,X = \frac{3}{2} - 1^2 = \frac{1}{2}$