Name _____

- Show work.

1. (16 pts) Suppose X is a discrete random variable with the pdf
$$P(X = k) = p_X(k) = \theta(1 - \theta)^k \quad \text{for } k = 0, 1, 2, 3, \dots$$
where θ is an unknown parameter. Find the maximum likelihood estimate (**MLE**) for θ based on a random sample of size n: $k_1, \dots, k_n$.

2. (12 pts) Suppose $E(X) = \theta/2$, $E(Y) = \theta$, and $Var(X) = Var(Y) = \theta$. Consider the estimator
$$\hat{\theta} = 2cX + (1 - c)Y \qquad \text{with } 0 < c < 1$$

a) (5 pts) Show that $\hat{\theta}$ is unbiased for all values of c.

b) (7 pts) Find the value of c which minimizes the variance of $\hat{\theta}$.

3. (12 pts) A roulette wheel is supposed to land on red 9 out of 19 times (or 47.37%). Fred found that the roulette wheel he recently bought landed on red 305 times in 600 spins.

    a) (7 pts) Construct a 95% confidence interval for the proportion of landing on red for Fred's roulette wheel, and assess if the wheel is working properly.

    b) (5 pts) If he wants the margin of error to be 1%, how many spins should he make?

4. (15 pts) Suppose you collected a sample of 100 data points from a normal distribution with unknown mean μ and known $\sigma = 2.4$, and calculated a 95% Z confidence interval.

    a) (5 pts) What is the width of the confidence interval?

    b) (5 pts) If two independent samples of size 100 are collected and two 95% intervals are constructed, what is the probability that only one of intervals contain the true mean?

    c) (5 pts) Suppose we want to keep the confidence level at 95% but halve the width, what is the sample size we should have?

5. (13 pts) A researcher claims that 25% of the U.S. population has circulation problems, but a rival researcher claims that there may be evidence in her study that the percentage should be lower, according to a random sample of 189 persons, where 38 are found to have circulation problems.

    a) (8 pts) State your null and alternative hypothesis. What do you conclude at the $\alpha = 0.05$ significance level about the claim of the rival research? Use the P-value.

    b) (5 pts) For what values of α will the null hypothesis be rejected?

6.   (12 pts) Suppose we test an unknown proportion p with $H_0: p = 0.3$ against $H_1: p > 0.3$ at $\alpha = 0.1$. The sample size is 500.

   a)   (6 pts) If the number of success is 161, do we reject H_0? Use the test statistic.

   b)   (6 pts) What is the smallest number of successes that will cause H_0 to be rejected?

7.   (6 pts) We test the null hypothesis $H_0 : \mu = 25$ against $H_1 : \mu \neq 25$ based on a random sample of size $n = 30$ drawn from a normal distribution with $\sigma = 8$. The decision rule is to reject H_0 when $\bar{x} < 23$ or $\bar{x} > 27$, where $\bar{x}$ is the sample mean. Find the significance level α of the test.

8.   (14 pts) We test the null hypothesis $H_0 : \mu = 105$ against $H_1 : \mu > 105$ based on a random sample of size $n = 55$ drawn from a normal distribution with $\sigma = 18$ at the significance level $\alpha = 0.01$.

   a)   (5 pts) Find the critical value.

   b)   (9 pts) Find the probability of committing a Type II error **and** the power of the test if the true mean is 112.