1. (8 pts) A box contains 3 number cards, with 1, 2, 3 on them. One card is drawn. Let X be the number on that card. A second card is drawn from the remaining two cards in the box. Let Y be the second card which is drawn. Write down the joint pdf for X and Y as a 2-way table.

$Y\backslash X$	1	2	3
1	0	1/6	1/6
2	1/6	0	1/6
3	1/6	1/6	0

2. (12 pts) In an exercise, we found the following joint pdf, where X is the number of heads on the last flip, and Y is the total number of heads in on 3 flips of a fair coin.

$X\backslash Y$	0	1	2	3	
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
	1/8	3/8	3/8	1/8	

Blue row: marginal pdf for Y Yellow column: marginal pdf for X

a) (4 pts) Find the marginal pdf's $p_X(x)$ and $p_Y(y)$, you may write them on the margin.

b) (8 pts) Calculate the **covariance** of X and Y. Explain why its sign makes sense.

$$E(X) = \frac{1}{2}, \qquad E(Y) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$E(XY) = 1(1)\left(\frac{1}{8}\right) + 1(2)\left(\frac{1}{4}\right) + 1(3)\left(\frac{1}{8}\right) = 1$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 1 - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{1}{4}$$

X and Y are positively correlated. This makes sense because if the last flip is a head, it is more likely to have more heads in total.

3. (8 pts) Find c, given that the pdf of X is

$$f_X(x) = c(2-x), \qquad 0 < x \le 1$$

$$\int_0^1 c(1-x)\,dx = c\left[2 - \frac{x^2}{2}\right]_0^1 = c\left(2 - \frac{1}{2}\right) = \frac{3}{2}c = 1 \quad \Rightarrow \quad c = \frac{2}{3}$$

4. (8 pts) Find the cdf of X, given that the pdf of X is

$$f_X(x) = \begin{cases} -x & -1/\le x \le 0 \\ x & 0 < x \le 1 \end{cases}$$

For $-1 \le x \le 0$,

$$F_X(x) = \int_{-1}^{x} -t\,dt = \left[-\frac{t^2}{2}\right]_{-1}^{x} = \frac{1}{2}(1-x^2)$$

For $0 \le x \le 1$,

$$F_X(x) = \int_{-1}^{0} -t\,dt + \int_0^x t\,dt = \frac{1}{2} + \left[\frac{t^2}{2}\right]_0^x = \frac{1}{2}(1+x^2)$$

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{2}(1-x^2), & -1 \le x \le 0 \\ \frac{1}{2}(1-x^2), & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

5. (8 pts) The cdf of X is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

a) (4 pts) Find $P(1/3 \le x \le 1/2)$ from the cdf.

$$P\left(\frac{1}{3} \le x \le \frac{1}{2}\right) = F_X\left(\frac{1}{2}\right) - F_X\left(\frac{1}{3}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$$

b) (4 pts) Find the pdf of X.

$$f_X(x) = F_X'(x) = 2x, \qquad 0 \le x \le 1$$

6. (8 pts) The pdf of X is

$$f_X(x) = 4xe^{-2x}, \qquad x > 0$$

The following formula is useful for this problem.

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

a) (3 pts) Find $E(X)$.

$$E(X) = \int_0^\infty 4x^2 e^{-2x}\,dx = 4\left(\frac{2!}{2^{2+1}}\right) = 1$$

b) (5 pts) Find $Var(X)$.

$$E(X^2) = \int_0^\infty 4x^3 e^{-2x}\,dx = 4\left(\frac{3!}{2^{3+1}}\right) = \frac{3}{2}$$

$$Var(X) = \frac{3}{2} - 1^2 = \frac{1}{2}$$

7. (8 pts) The joint pdf of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{3} + xy, \qquad 0 \le x \le 1, \qquad 0 \le y \le 2$$

a) (4 pts) Find $f_X(x)$.

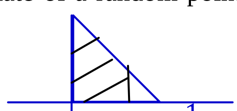
$$f_X(x) = \int_0^2 \left(\frac{1}{3} + xy\right) dy = \left[\frac{y}{3} + \frac{xy^2}{2}\right]_0^2 = \frac{2}{3} + 2x, \qquad 0 \le x \le 1$$

b) (4 pts) Set up a double integral (complete with limits) to find $P(Y \ge 2X)$, but do **NOT** evaluate it.

$$P(Y \ge 2X) = \int_0^1 \int_{2x}^2 \left(\frac{1}{3} + xy\right) dy\,dx$$

Note for #7: I didn't choose the joint pdf carefully, so $\int_0^1 \int_0^2 f_{X,Y}\,dydx \ne 1$, it didn't affect what you should do to answer the question, but sorry about that, especially for students who checked if f_X integrates to 1, and it did not!

8. (6 pts) Let (X, Y) be the coordinate of a random point picked from the triangle with vertices $(0,0), (3,0), (0,3)$. Find $P(X \le 2)$.



$$P(X \le 2) = 1 - \frac{\frac{1}{2}(1)(1)}{\frac{1}{2}(3)(3)} = 1 - \frac{1}{9} = \frac{8}{9}$$

(**Extra credit 3 pts**) Without calculation, give the sign of $\mathrm{Cov}(X, Y)$ and explain how you know.

$\mathrm{Cov}(X, Y)$ is negative because as X increases, the range of Y decreases and Y tends to decrease.

9. (6 pts) The number of major snowstorms has a Poisson distribution with rate 0.4 per month from January through March. Find the probability there are 2 or fewer snow storms during the 3-month period.

$$P(X \le 2) = e^{-1.2} + 1.2e^{-1.2} + \frac{1.2^2 e^{-1.2}}{2!} = .8795$$

10. (6 pts) In a class of 100 students, use **Poisson approximation** to find the probability that at least one person is born on July 4th. Assume that there are 365 days in every year.

$$\lambda = np = \frac{100}{365}, \qquad 1 - P(X = 0) = 1 - e^{-100/365} = .2396$$

11. (6 pts) The volume of a bottle of soft drink is normally distributed with mean 20 oz and standard deviation 1.5 oz. What is the value x such that 95% of the bottles have at least x oz?

$$P(Vol \ge x) = 0.95 \quad \Rightarrow \quad x = invNorm(0.05, 20, 1.5) = 17.53$$

12. (9 pts) Suppose the height of people in Country A has a normal distribution with mean 5.4 and standard deviation 0.3, while the height of people in Country B has a normal distribution with mean 5.8 and standard deviation 0.6. What is the probability that a randomly person in Country A is taller than a random person in Country B?

Let X be the height of a person from Country A and Y be the height of a person from Country B, and $W = X - Y$

$$E(W) = E(X) - E(Y) = 5.4 - 5.8 = -0.4$$

$$Var(W) = Var(X) + Var(Y) = 0.3^2 + 0.6^2 = 0.09 + 0.36 = 0.45$$

$$W \sim Normal(-0.4, 0.45)$$

$$P(X > Y) = P(W > 0) = normcdf\left(0, \infty, -0.4, \sqrt{0.45}\right) = .2755$$

13. (9 pts) The weekly repair cost for a certain machine is a random variable with mean \$95 and standard deviation \$25. Suppose the budget for repair is \$5200 for a year (52 weeks). Use Central Limit Theorem to estimate the probability that the actual repair cost exceeds the budgeted amount.

Let X be the repair cost for 52 weeks, then $E(X) = 52 \times 95$, $Var(X) = 52 \times 25^2$, and $X \sim Normal(4940, 32500)$

$$P(X \ge 5200) = normcdf\left(5200, \infty, 4940, \sqrt{32500}\right) = 0.0746$$