Page	1	2	3	4	5	6	7	8	Total
Points	11	13	14	14	10	14	10	14	100

Math 3081 Spring 2018 **Final Exam**

NAME:__________________ **Instructor:**__________________

Show your work!

1. Let the continuous random variable X have pdf $f_X(x) = c(x(x+1))$ for $1 \le x \le 2$ Find:

a) (3 points) c.

$$1 = c\int_1^2 x^2 + x \, dx = c\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_1^2 = \frac{23}{6}c$$

$$\rightarrow \underline{c = \frac{6}{23}}$$

b) (4 points) E(X).

$$= \frac{6}{23}\int_1^2 x\,(x^2+x)\,dx = \frac{6}{23}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\Big|_1^2 = \frac{6}{23}\left(\frac{73}{12}\right)$$

$$= \underline{\frac{73}{46} = 1.587}$$

c) (4 points) Var(X).

$$E(X^2) = \frac{6}{23}\int_1^2 x^2\,(x^2+x)\,dx = \frac{6}{23}\left(\frac{1}{5}x^5 + \frac{1}{4}x^4\right)\Big|_1^2$$

$$= \frac{6}{23}(9.95) = 2.596$$

$$Var(X) = E(X^2) - (E(X))^2 = 2.596 - \left(\frac{73}{46}\right)^2 = \underline{.0772}$$

2. (4 points) You roll a fair die twice. If X=|(roll on die 1)−(roll on die 2)|, then give the probability distribution for X.

$X = 0, 1, 2, 3, 4, 5, 1, 0, 1, \ldots$

$S_x = \{0, 1, 2, 3, 4, 5\}$, $[S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, \ldots\}$

$P(X=0) = 6/36$ (if $\{11, 22, 33, 44, 55, 66\}$)

$P(X=1) = 10/36$ (if $\{12, 21, 23, 32, 34, 43, 45, 54, 56, 65\}$)

$P(X=2) = 8/36$

$P(X=3) = 6/36$

$P(X=4) = 4/36$

$P(X=5) = 2/36$

3. (5 points) Given: P(A)+P(B)=1.2, $P(B|A) = .6$, and $P(A \cup B)$=.9, find P(A).

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = .6$$

$$.9 = P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1.2 - P(A \cap B)$$

$$\rightarrow P(A \cap B) = .3$$

$$\rightarrow P(A) = \frac{.3}{.6} = .5$$

4. (4 points) A sample of 23 from a normal distribution finds a mean of 109.7. If we know $\sigma = 12.2$, find a 99% confidence interval for the population mean.

$$z = 2.576 \rightarrow 109.7 \pm 2.576\left(\frac{12.2}{\sqrt{23}}\right) \rightarrow 109.7 \pm 6.55$$

or Z Interval $\rightarrow$ $(103.15, 116.25)$

5. (6 points) You roll a fair die 300 times. Estimate P(number of sixes $\leq$ 45) using the normal approximation with continuity correction.

mean $= 300\left(\frac{1}{6}\right) = 50$, $\sigma = \sqrt{300\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} = 6.455$

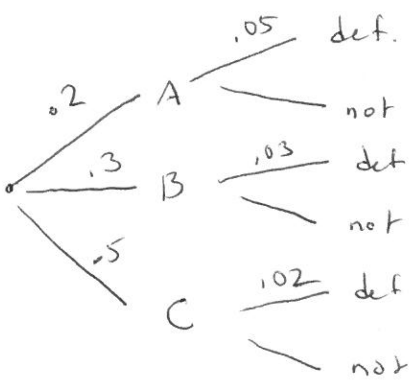
Cont. Corr. $\rightarrow$ $P(X \leq 45.5) \approx P\left(Z \leq \frac{45.5 - 50}{6.455} = -.7\right) = \underline{.242}$

or normalcdf $\left(-10000, 45.5, 50, \sqrt{300\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)}\right) = \underline{.2429}$

something really small

6. A company builds its product at 3 plants. After complaints, they find that: 5% of the product built at plant A are defective, 3% of the product built at plant B are defective, 2% of the product built at plant C are defective. 20% of the products were built at plant A, 30% at plant B, and the rest at plant C. If one of the product is chosen randomly, find:

a) (4 points) P(it's defective).



$= .2(.05) + .3(.03) + .5(.02)$

$= \underline{.029}$

b) (4 points) P(it's from plant C | it's defective).

$= \frac{P(C \cap \text{def})}{P(\text{def})} = \frac{.5(.02)}{.029} = \underline{\frac{10}{29}} = .345$

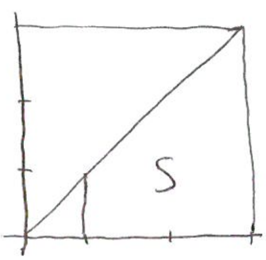
7. (4 points) Suppose an average of 20 meteorites hit the Earth each day on average. If the number of meteorites that hit has a Poisson distribution, find the probability that exactly four hit the Earth in the next **three hours**. Give your answer to three decimal places.

$$\lambda = \frac{20}{8} = \frac{5}{2} \rightarrow \frac{e^{-5/2}\left(\frac{5}{2}\right)^4}{4!} = \underline{.134}$$

or $\quad$ poissonpdf$(5/2, 4) =$

8. Given $f_{X,Y}(x,y) = \frac{1}{10}(xy)$ for: $y < x$, $1 < x < 3$, and $0 < y < 3$.

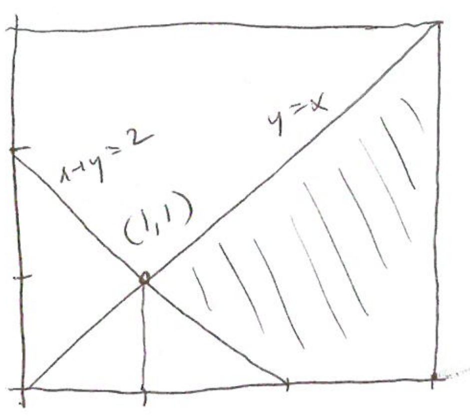
a) (6 points) Find E(X). Show your work.



$$= \frac{1}{10}\int_1^3 \int_0^x x(xy)\,dy\,dx = \frac{1}{10}\int_1^3 \frac{1}{2}x^2y^2 \Big|_0^x dx$$

$$= \frac{1}{10}\int_1^3 \frac{1}{2}x^4\,dx = \frac{1}{100}x^5 \Big|_1^3 = \underline{2.42}$$

b) (4 points) Find $P(X + Y > 2)$. Set up the integral only.



$$= \frac{1}{10}\left[\int_1^2 \int_{2-x}^x xy\,dy\,dx + \int_2^3 \int_0^x xy\,dy\,dx\right]$$

or $\quad 1 - \frac{1}{10}\int_1^2 \int_0^{2-x} xy\,dy\,dx$

or $\quad$ ---

9. (3 pts) We run a hypothesis test, on normally distributed data, with $H_0 : \mu = 5$ against $H_1 : \mu \neq 5$ at the 1% level of significance and find a test statistic of -2.7. If we use a sample of 20 and we use the sample standard deviation, then give the critical value and decide if you accept or reject H_0.

$df = 19, \ .01/2 \quad \rightarrow \quad t = \underline{2.861} = CV$

$\rightarrow$ $\underline{\text{fail to reject } H_0}$

10. (7 points) Suppose the random variables $X_1, X_2, \ldots X_n$ are independent each with the distribution

$f(x;\theta) = \dfrac{\theta 2^\theta}{x^{(\theta+1)}}$ for $x \geq 2$.

Find the Maximum Likelihood estimate for θ.

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta 2^\theta}{x_i^{\theta+1}} = \theta^n 2^{n\theta} \left(\prod_{i=1}^{n} x_i \right)^{-(\theta+1)}$$

$$\ln(L(\theta)) = n \ln \theta + n\theta \ln 2 - (\theta+1) \ln \left(\prod_{i=1}^{n} x_i \right)$$

$$\frac{d}{d\theta}(\text{" \quad "}) = \frac{n}{\theta} + n \ln 2 - \ln \left(\prod_{i=1}^{n} x_i \right) \overset{set}{=} 0$$

$$\rightarrow \quad \frac{n}{\theta} = \ln \left(\prod_{i=1}^{n} x_i \right) - n \ln 2$$

$$\rightarrow \quad \hat{\theta} = \frac{n}{\ln \left(\prod_{i=1}^{n} x_i \right) - n \ln 2}$$

$$\left(\text{or} \quad \frac{n}{\sum_{i=1}^{n} \ln(x_i) - n \ln 2} \right)$$

11. (6 points) A study in Lesotho tested whether offering same day antiretrovial treatment for HIV would change the percent of patients with viral suppression after 12 months. A sample of 137 patients offered same-day treatment found 69 achieved viral suppression, while 47 out of 137 patients in the usual-care group achieved viral suppression. Test at the 5% level of significance to see if the percents are the same.
You should give your null and alternate hypotheses, then decide if you accept or reject the null hypothesis (find the test statistic and either the p-value or critical value(s)).

$H_0: p_{sd} = p_{uc}$, $H_1: p_{sd} \neq p_{uc}$

$\hat{p}_c = \dfrac{47+69}{137+137} = .423$ $\hat{p}_{sd} = \dfrac{69}{137} = .504$, $\hat{p}_{uc} = .343$

$\rightarrow$ $z = \dfrac{.504 - .343}{\sqrt{.423(.577)\left(\frac{1}{137}+\frac{1}{137}\right)}} = \underline{2.69 = t.s.}$ (or 2.7)

$z^*, .025 \rightarrow \underline{z = 1.96 = c.v.} \rightarrow \underline{\text{reject } H_0}$

or 2-Prop Z Test $\rightarrow$ $z = 2.69$, p-value $= .007$ $\rightarrow$ reject H_0

12. (6 points) The mean internal body temperature of adults is supposed to be 37° C. A random sample of 25 adults who are living above the Arctic Circle finds a mean of 36.5° and a sample standard deviation of 1.3. Use a one-sided test to test at the 5% significance level to see if the mean of those living above the Arctic Circle is less than 37°.
You should give your null and alternate hypotheses, then decide if you accept or reject the null hypothesis (find the test statistic and either the p-value or critical value(s)).

$H_0: \mu = 37$, $H_1: \mu < 37$

$t = \dfrac{36.5 - 37}{1.3/\sqrt{25}} = \underline{-1.92 = t.s.}$

$df = 24, .05 \rightarrow \underline{t = 1.711 = c.v.}$

reject H_0

or T-test $\rightarrow$ $t = -1.923$, p-value $= .033$ $\rightarrow$ reject H_0

13. (2 points) We test $H_0: \mu = 10$ against $H_1: \mu > 10$ at the 5% level and reject H_0. If we build a 95% confidence interval, would it contain 10? Possible answers are: Yes, No, Can't tell.

p-value $< .05$ for 1-tailed test $\rightarrow$ p-value $< .1$ for 2-tailed

$\rightarrow$ Can't tell

14. (4 points) A company wants to know the mean age of its customers. If they want a margin of error of at most 2 at the 99% confidence level, then how large a sample should they look at if they know $\sigma = 14$?

$$z = 2.576 \rightarrow n \geq \left(\frac{2.576(14)}{2} \right)^2 = 325.15$$

$$\rightarrow \underline{n = 326}$$

15. (6 points) A sample of 14 found the mean age for Boston was 31.1 with a standard deviation of 7.8; a sample of 15 found the mean age in NY was 35.9 with s=8.6. Test at the 5% level to see if the mean age is different in the two cities. Use the **pooled** test.

You should give your null and alternate hypotheses, then decide if you accept or reject the null hypothesis (find the test statistic and either the p-value or critical value(s)).

$H_0: \mu_B = \mu_{NY}$, $H_1: \mu_B \neq \mu_{NY}$

$$s_p = \sqrt{\frac{13(7.8)^2 + 14(8.6)^2}{27}} = 8.22$$

$$t = \frac{31.1 - 35.9}{8.22\sqrt{\frac{1}{14} + \frac{1}{15}}} = \underline{-1.57 = \text{t.s.}}$$

$df = 27, .025 \rightarrow t = \underline{2.052} = \text{c.v.}$

$\Big\}$ <u>fail to reject Ho</u>

or

2- SampTTest $\rightarrow$ $t = -1.57$, p-value $= .128 \rightarrow$ fail to reject Ho

16. A researcher is checking to see if a diet works. They will look at a sample of 100 and assume $\sigma = 12$. If they are testing $H_0 : \mu = 150$ against $H_1 : \mu < 150$ at the 1% level, then:
a) (5 points) find the critical values and give a rejection test.

$.01 \rightarrow z = 2.326$

$$\rightarrow -2.326 = \frac{\bar{x} - 150}{12/\sqrt{100}} \rightarrow \bar{x} = 147.21$$

$\rightarrow$ reject H_0 if $\bar{x} < 147.21$

b) (4 points) Find the power of the test if the real mean is 146.

$$= P(\bar{x} < 147.21, \mu = 146) = P\left(z < \frac{147.21 - 146}{12/\sqrt{100}} = 1.01\right)$$

$= .8438$

or normalcdf $(-10000, 147.21, 146, 12/\sqrt{100}) = .8434$

$(.8434$ using $\bar{x} = 147.2088)$

17. Suppose the random variables X and Y are independent with an unknown parameter θ. If E(X)=θ, E(Y)=2θ, Var(X)=θ^2, and Var(Y)=$2\theta^2$ then if W=aX+bY:
a) (3 points) what has to be true about the constants a and b so that W is an unbiased estimator for θ?

We need $\theta = E(W) = aE(X) + bE(Y) = a\theta + 2b\theta$

$\rightarrow a + 2b = 1$

b) (2 points) what is var(W)?

$= a^2 Var(X) + b^2 Var(Y) = a^2\theta^2 + 2b^2\theta^2$