Math 3081 Fall 2016 **Final Exam** NAME: Answers

Show your work!

1. Let the continuous random variable X have pdf $f_X(x) = 6x(1-x)$ for $0 \le x \le 1$.

a) (3 points) Find $E(X)$.

$= \int_0^1 x\,(6x(1-x))\,dx = 6\int_0^1 x^2 - x^3\,dx$

$= 6\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\Big|_0^1\right) = \underline{1/2}$

b) (4 points) Find $Var(X)$.

$E(X^2) = \int_0^1 x^2\,(6x(1-x))\,dx = 6\int_0^1 x^3 - x^4\,dx = 6\left(\frac{1}{4}x^4 - \frac{1}{5}x^5\right)\Big|_0^1$

$= .3$

$Var(X) = .3 - (1/2)^2 = \underline{.05}$

c) (4 points) Find $P(X < 1/4)$.

$= 6\int_0^{1/4} x - x^2\,dx = 6\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_0^{1/4}\right)$

$= \underline{5/32} = .15625$

d) (4 points) Pick three points in $0 \le x \le 1$; what is the probability that exactly two of them are less than 1/4?

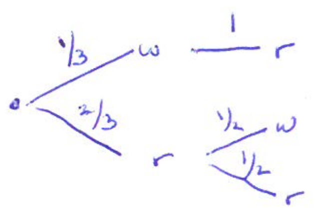
$P(X=2) = \binom{3}{2}\left(\frac{5}{32}\right)^2\left(\frac{27}{32}\right) = \underline{.062}$

2. (3 points) We run a hypothesis test with $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu \neq \mu_2$ and find a test statistic of -2.8. If we know the critical value is 2.2 then decide if you accept or reject H_0.

reject H_0

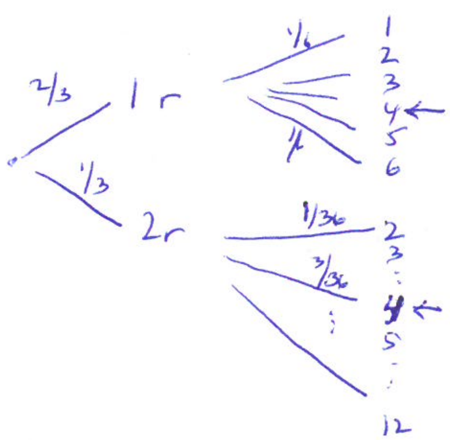
3. You have an urn with 1 white chip and 2 red chips. Draw 2 chips without replacement from the urn:
If you draw 1 red chip, then roll one fair die and let S be the number (between 1 and 6) shown.
If you draw 2 red chips, roll two fair dice and let S be the sum of the two numbers shown.

a) (6 points) Find $P(S = 4)$



$\rightarrow P(1\,r) = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{2}\right) = 2/3$

$P(2\,r) = 1/3$



$P(S=4) = \frac{2}{3}\left(\frac{1}{6}\right) + \frac{1}{3}\left(\frac{3}{36}\right)$

$= \underline{5/36} = .13\overline{8}$

b) (4 points) If $S = 4$, find the probability that you drew 2 red chips.

$= P(2r \mid S=4) = \dfrac{P(2r \cap S=4)}{P(S=4)}$

$= \dfrac{\frac{1}{3}\left(\frac{3}{36}\right)}{5/36} = \underline{1/5}$

4. (6 points) A fair coin is tossed 100 times. Estimate the probability that Heads come up at least 60 times. (Include a continuity correction in your estimate.)

$$\text{mean} = 100\left(\tfrac{1}{2}\right) = 50, \quad \sigma = \sqrt{100\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)} = 5$$

$$P(X \geq 60) \rightarrow P(X \geq 59.5) \approx P\left(Z \geq \frac{59.5-50}{5} = 1.9\right)$$

$$= 1 - .9713 = \underline{.0287}$$

5. (4 points) The IQ scores of the public are normally distributed with a mean of 100 and standard deviation of 15. If 10 people are chosen at random, what is the probability that their average IQ exceeds 110?

$$P(\bar{X} > 110) = P\left(Z > \frac{110-100}{15/\sqrt{10}}\right) = P(Z > 2.11)$$

$$= 1 - .9826 = \underline{.0174}$$

$$(\text{calc.} \rightarrow .0175)$$

6. (4 points) A sample of 15 finds a mean of 19.4 and a sample standard deviation of 6.5. Find a 95% confidence interval for the population mean.

$$df = 14, \rightarrow t = 2.145$$

$$\rightarrow 19.4 \pm 2.145\left(\frac{6.5}{\sqrt{15}}\right)$$

$$\rightarrow \underline{19.4 \pm 3.6}$$

or $(15.8, 23)$

7. (4 points) The number of customers that come to a certain bank on a Tuesday is a Poisson random variable with a mean of 12 customers every 20 minutes. Find the probability of getting exactly 5 customers from 11:00am to 11:05am. Give your answer to three decimal places.

$$\lambda = 12/4 = 3 / 5\text{min}$$

$$P(X=5) = \frac{e^{-3}3^5}{5!} = \underline{.101}$$

8. Suppose $f_{X,Y}(x,y) = \frac{1}{3}(x+y)$ for $0 \le x \le 2$ and $0 \le y \le 1$.

a) (4 points) Find $E(X)$.

$$= \frac{1}{3}\int_0^2 \int_0^1 x(x+y)\,dy\,dx$$

note: This gives $f_X(x) = \frac{1}{3}\left(x + \frac{1}{2}\right)$

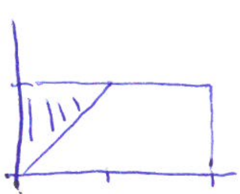
$$= \frac{1}{3}\int_0^2 x^2 y + \frac{1}{2}x^2 y \Big|_0^1 dx = \frac{1}{3}\int_0^2 x^2 + \frac{1}{2}x\,dx$$

$$= \frac{1}{3}\left(\frac{1}{3}x^3 + \frac{1}{4}x^2 \Big|_0^2\right) = \underline{11/9} = 1.\overline{2}$$

(Note: they can also get $f_X(x)$ first: $f_X(x) = \frac{1}{3}\int_0^1 x+y\,dy =$

then $E(X) = \int_0^2 x\, f_X(x)\,dx$ ...)

b) (5 points) Find $P(X < Y)$.



$$= \frac{1}{3}\int_0^1 \int_x^1 x+y\,dy\,dx$$

$$= \frac{1}{3}\int_0^1 xy + \frac{1}{2}y^2 \Big|_x^1 dx$$

$$= \frac{1}{3}\int_0^1 x + \frac{1}{2} - x^2 - \frac{1}{2}x^2\,dx$$

$$= \frac{1}{3}\int_0^1 x + \frac{1}{2} - \frac{3}{2}x^2\,dx = \frac{1}{3}\left(\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{2}x^3 \Big|_0^1\right)$$

$$= \underline{1/6}$$

9. (8 points) Suppose the random variables $X_1, X_2, \ldots X_n$ are independent each with the distribution $f(x; b) = 2bx(1-x^2)^{b-1}$ for $0 \le x \le 1$.

Find the Maximum Likelihood estimate for b.

$$L(b) = \prod_{i=1}^{n} 2 b x_i (1 - x_i^2)^{b-1}$$

$$= 2^n b^n \left(\prod_{i=1}^{n} x_i \right) \left(\prod_{i=1}^{n} (1 - x_i^2) \right)^{b-1}$$

$$\ln(L(b)) = n \ln 2 + n \ln b + \ln \left(\prod_{i=1}^{n} x_i \right) + (b-1) \ln \left(\prod_{i=1}^{n} (1-x_i^2) \right)$$

$$\frac{d}{db}\left('' \quad '' \right) = \frac{n}{b} + \ln \left(\prod_{i=1}^{n} (1 - x_i^2) \right) \overset{\text{set}}{=} 0$$

$$\rightarrow \quad \hat{b} = \frac{-n}{\ln \left(\prod_{i=1}^{n} (1 - x_i^2) \right)}$$

(note: they can also use $\ln \left(\prod_{i=1}^{n} (1 - x_i^2) \right) = \sum_{i=1}^{n} \ln(1 - x_i^2)$)

10. Given the samples:

Sample A: $n = 200, \hat{p} = .3$
Sample B: $n = 300, \hat{p} = .25$

a) (5 points) find a 95% confidence interval for the difference in the percents.

$z = 1.96$

$\rightarrow \quad (.3 - .25) \pm 1.96\sqrt{\frac{.3(.7)}{200} + \frac{.25(.75)}{300}}$

$\rightarrow \quad .05 \pm .08$

or $\quad (-.03, .13)$

b) (2 points) if we test $H_0 : p_A = p_B$ against $H_1 : p_A \neq p_B$ at the 5% level, then do we accept or reject H_0? Use your answer from part (a).

accept or fail to reject H_0

11. (6 points) A coin is checked to see if it's fair. In a sample of 300 tosses, there are 165 tails. Test at the 5% significance level to see if the percent of tails is 50% or if it's more.
You should give your null and alternate hypotheses, then decide if you accept or reject the null hypothesis (find the test statistic and either the p-value or critical value(s)).

$H_0 : p = .5 \quad , \quad H_1 : p > .5$

$\hat{p} = .55$

$z = \dfrac{.55 - .5}{\sqrt{\frac{.5(.5)}{300}}} = 1.73 = t.s.$

either

(i) $z^*, .05 \rightarrow z = 1.645 = c.v.$

(ii) 1-PropZTest $\rightarrow$ p-value $= .042$

$\Big\}$ reject H_0

12. (4 points) A company wants to estimate the mean time it takes to complete a task. If they want a margin of error of at most 2 minutes at the 90% confidence level and use $\sigma = 25$ minutes, then how large a sample should they look at?

$z = 1.645 \quad \rightarrow \quad n \geq \left(\frac{1.645(25)}{2} \right)^2 = 422.82$

$\rightarrow \quad \underline{n = 423}$

$(z = 1.65 \rightarrow n = 426 \quad , \quad z = 1.64 \rightarrow n = 421)$

13. (7 points) A study looked to see how patients with stem cell transplants responded to the palliative versus standard care. 17 patients were given standard care and had a mean of 86.6 and a standard deviation of 10.8 on a quality of life measure; 14 patients were given palliative care and had a mean of 94.33 and a standard deviation of 11.6 on a quality of life measure. Test at the 5% level to see if there is a difference between means of the two groups. You can assume the variances are the same.

You should give your null and alternate hypotheses, then decide if you accept or reject the null hypothesis (find the test statistic and either the p-value or critical value(s)).

$H_0: \mu_S = \mu_P \quad , \quad H_1: \mu_S \neq \mu_P$

$$s_p = \sqrt{\frac{16(10.8)^2 + 13(11.6)^2}{29}} = 11.166$$

$$t = \frac{86.6 - 94.33}{11.166\sqrt{\frac{1}{17} + \frac{1}{14}}} = \underline{-1.92 = t.s}$$

<u>either</u> (1) $df = 29, .025 \rightarrow t = \underline{2.06 = C.V.}$

(2) 2-SampTTest $\rightarrow$ <u>p-value = .065</u>

$\Big\}$ <u>fail to reject H_0</u>

$\Big[$ non-pooled $\rightarrow$ $t = \dfrac{86.6 - 94.33}{\sqrt{\frac{10.8^2}{17} + \frac{11.6^2}{14}}} = -1.9$, p-value = .0675

CV: $df = 13, \rightarrow t = 2.16$ $\Big]$

14. A researcher wants to see if the average commute time in Boston is increasing. They will look at a sample of 400 and assume $\sigma = 19$ minutes. If they are testing at the 5% level of significance and use $H_0 : \mu = 27.3$ and $H_1 : \mu > 27.3$, then:

(a) (5 points) find the critical values and use it to get a rejection test.

$$z = 1.645 \rightarrow 1.645 = \frac{\bar{x} - 27.3}{19/\sqrt{400}} \rightarrow \bar{x} = 28.863$$

$\rightarrow$ <u>reject H_0 if $\bar{x} > 28.863$</u>

$(1.65 \rightarrow \bar{x} = 28.8675, \quad 1.64 \rightarrow \bar{x} = 28.858)$

(b) (4 points) find the power if the real mean is 30.

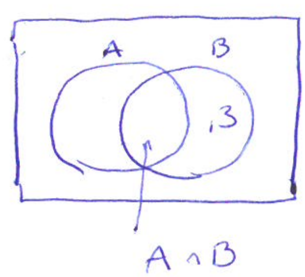
$$= P(\bar{X} > 28.863, \mu = 30) = P\left(Z > \frac{28.863 - 30}{19/\sqrt{400}} = -1.2\right)$$

$$= 1 - .1151 = \underline{.8849} \quad (\text{calc.} \rightarrow .8844)$$

$(1.65 \rightarrow z = -1.19 \rightarrow .883, \quad 1.64$ also gives $.8849)$

15. (4 points) Suppose A and B are events such that $P(A|B) = .4$ and $P(A^c \cap B) = .3$. Find $P(B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = .4$$



$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$= .4\,P(B) + .3$$

$$\rightarrow .6\,P(B) = .3 \rightarrow \underline{P(B) = .5}$$