Math 4571: Advanced Linear Algebra

Practice Midterm 2B (Instructor: Dummit)

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 6 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	16	
2	12	
3	8	
4	12	
5	12	
6	12	
7	8	
TOTAL	80	

1. (16 points) For each of the following, circle the correct response (there is no partial credit nor penalty for wrong answers).

True **False** The Cauchy-Schwarz inequality is true in $\mathbb{R}^n$ but false in any other space.

True **False** In any inner product space, $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.

True **False** Every finite-dimensional inner product space has an orthonormal basis.

True **False** Every linear transformation has an adjoint.

True **False** The vector $\mathbf{v} = e^{-x} - 2e^{x}$ is an eigenvector for the map $D^2 : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $D^2(f) = f''$.

True **False** If two matrices are similar, then they have the same eigenvalues.

True **False** A matrix is invertible if and only if it does not have 0 as an eigenvalue.

True **False** If the characteristic polynomial for A is $(t-1)^4(t-2)^2$, then the determinant of A is 2.

True **False** Any $n \times n$ matrix having only one eigenvalue is never diagonalizable.

True **False** If A is any non-diagonalizable matrix having characteristic polynomial p, then $p(A)$ is the zero matrix.

True **False** If two matrices have the same Jordan canonical form, then they are similar.

True **False** Real symmetric matrices are diagonalizable.

2. (12 points) Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a set of vectors in the real inner product space V, where $||\mathbf{u}_i|| = 1$ for each $1 \le i \le n$.

(a) If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal set, show $||\mathbf{u}_i + \mathbf{u}_j|| = \sqrt{2}$ for every $1 \le i < j \le n$.

(b) If $||\mathbf{u}_i + \mathbf{u}_j|| = \sqrt{2}$ for every $1 \le i < j \le n$, show $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal set.

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3. (8 points) Prove $\sqrt{a^2 + 2ab + 3b^2} + \sqrt{c^2 + 2cd + 3d^2} \le \sqrt{(a+c)^2 + 2(a+c)(b+d) + 3(b+d)^2}$ for all real a, b, c, d.

4. (12 points) Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \to V$ is a linear transformation possessing an adjoint T^*. Recall that $S : V \to V$ is Hermitian when $S = S^*$.

(a) Show that $T + T^*$, $i(T - T^*)$, T^*T, and TT^* are all Hermitian.

(b) Prove that $\ker(T^*T) = \ker(T)$.

(c) Suppose T is Hermitian. Show that $\langle T(\mathbf{v}), \mathbf{v}\rangle$ is a real number for any vector $\mathbf{v}$.

5. (12 points) Suppose V is a finite-dimensional inner product space and $T : V \to V$ is linear.

(a) If $\langle T(\mathbf{v}), \mathbf{w} \rangle = 0$ for all $\mathbf{w}$ in V, show that $T(\mathbf{v}) = \mathbf{0}$.

(b) If T is one-to-one, show that T^* is onto.

(c) If T is onto, show that T^* is one-to-one.

6. (12 points) We say $B \in M_{n\times n}(\mathbb{C})$ is a square root of $A \in M_{n\times n}(\mathbb{C})$ if $B^2 = A$.

(a) If A is diagonalizable, show that it has a square root.

(b) Show that the non-diagonalizable matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ does not have a square root.
[Hint: If $B^2 = A$, show $B^4 = 0$ and then consider the characteristic polynomial of B.]

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7. (8 points) Let $D : P_3(\mathbb{C}) \to P_3(\mathbb{C})$ be the derivative map. Show that D is not diagonalizable, and then find the Jordan canonical form of D.