Math 4571: Advanced Linear Algebra

Practice Midterm 2A (Instructor: Dummit)

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 6 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	16	
2	12	
3	8	
4	10	
5	16	
6	8	
7	10	
TOTAL	80	

1. (16 points) For each of the following, circle the correct response (there is no partial credit nor penalty for wrong answers).

True **False** The vector space $\mathbb{R}^4$ has exactly one inner product.

True **False** In any inner product space, $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2 \rangle = \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_2 + \mathbf{w}_2 \rangle$.

True **False** In any inner product space, $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq ||\mathbf{v}|| \, ||\mathbf{w}||$.

True **False** The set $\frac{1}{3}(1, 2, 2)$, $\frac{1}{\sqrt{5}}(0, 1, -1)$, $\frac{1}{\sqrt{18}}(-4, 1, 1)$ is an orthonormal basis for $\mathbb{R}^3$, under the dot product.

True **False** If S^* and T^* exist, then $(S + iT)^* = S^* + iT^*$.

True **False** If two matrices have the same eigenvalues, then they are similar.

True **False** If two matrices are similar, then they have the same eigenvectors.

True **False** If the characteristic polynomial for A is $(t-1)^4(t-2)^2$, then the 1-eigenspace of A is 4-dimensional.

True **False** Any $n \times n$ matrix having only one eigenvalue is diagonalizable.

True **False** The only $n \times n$ matrix A with $(A - I_n)^n = \mathbf{0}$ is the identity matrix.

True **False** If J_1 and J_2 are two matrices in Jordan canonical form and J_1 is similar to J_2, then $J_1 = J_2$.

True **False** If two matrices are similar, then they have the same Jordan canonical form.

2. (12 points) For each of the following pairings $\langle \cdot, \cdot \rangle$ on the vector space V, determine with justification whether or not it is an inner product on V:

(a) $V = C[0,1]$, with $\langle f, g \rangle = f'(0)g(1)$.

(b) $V = \mathbb{R}^2$, with $\langle (a,b), (c,d) \rangle = 4ac + ad + bc + 4bd$.

(c) $V = P_3(\mathbb{C})$, with $\langle p, q \rangle = \int_0^1 p(x)\overline{q(x)}\, dx$.

3. (8 points) Suppose V is an inner product space with subspaces W_1 and W_2. Show $W_1^{\perp} + W_2^{\perp} \subseteq (W_1 \cap W_2)^{\perp}$.

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4. (10 points) Suppose that $T : V \to V$ is a linear transformation on a finite-dimensional inner product space.

(a) Show that all eigenvalues of T^*T are nonnegative real numbers.

(b) If B is any square complex matrix, show that $\det(I + B^*B)$ is a positive real number.

5. (16 points) Suppose V is a real vector space and that $T : V \to V$ has the property that $T^2 = I$.

(a) Show that the only possible eigenvalues of T are 1 and -1.

(b) Suppose that $(T - I)^2\mathbf{v} = \mathbf{0}$. Show in fact that $(T - I)\mathbf{v} = \mathbf{0}$.

(c) Suppose that $(T + I)^2\mathbf{v} = \mathbf{0}$. Show in fact that $(T + I)\mathbf{v} = \mathbf{0}$.

(d) Show that any generalized eigenvector of T is actually an eigenvector of T.

(e) If V is finite-dimensional, show that T is diagonalizable.

6. (8 points) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $T : V \to V$ with respective eigenvalues λ_1 and λ_2, and $\mathbf{v}_1 + \mathbf{v}_2 \neq \mathbf{0}$. Show that $\mathbf{v}_1 + \mathbf{v}_2$ is also an eigenvector of T if and only if $\lambda_1 = \lambda_2$.

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7. (10 points) Suppose that A is an $n \times n$ complex matrix of rank 1.

(a) Up to equivalence, what are the possible Jordan canonical forms of A?

(b) For each matrix you listed in part (a), what is its characteristic polynomial?