Math 4571: Advanced Linear Algebra

Practice Final B (Instructor: Dummit)

NAME (please print legibly): ____

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 10 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____

QUESTION	VALUE	SCORE
1	16	
2	8	
3	10	
4	12	
5	12	
6	10	
7	8	
8	8	
9	12	
10	16	
11	8	
TOTAL	120	

1. (16 points) For each of the following, circle the correct response (there is no partial credit nor penalty for wrong answers).

True **False** The vectors $\langle 1,1\rangle$, $\langle 4,1\rangle$, $\langle 2,3\rangle$ span $\mathbb{R}^2$.

True **False** The set of vectors in any vector space V forms a basis for V.

True **False** If for any $\mathbf{w}$ in W there is a unique $\mathbf{v}$ in V with $T(\mathbf{v}) = \mathbf{w}$, then T is an isomorphism.

True **False** If $\dim(\text{im}(T)) = \dim(W)$, then T is onto.

True **False** In any inner product space, $|\langle \mathbf{v}, \mathbf{w}\rangle| \le ||\mathbf{v}||\,||\mathbf{w}||$.

True **False** If two matrices have the same eigenvalues, then they are similar.

True **False** If the characteristic polynomial for A is $(t-1)^4(t-2)^2$, then the 1-eigenspace of A is 4-dimensional.

True **False** If two matrices have the same Jordan canonical form, then they are similar.

True **False** If $V = M_{n\times n}(F)$ and $\Phi(A,B) = \det(AB)$, then Φ is a bilinear form on V.

True **False** Every quadratic form on $\mathbb{R}^n$ is diagonalizable.

True **False** Congruent matrices have the same eigenvalues.

True **False** Every matrix $A \in M_{n\times n}(\mathbb{C})$ can be written as a product $A = PDQ$ where P and Q are unitary and D is diagonal.

2. (8 points) Suppose A and B are $n \times n$ matrices such that B and ABA are invertible. Show that A is invertible.

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3. (10 points) Suppose V and W are finite-dimensional vector spaces and $T : V \to W$ is linear, β is an ordered basis of V, and γ is an ordered basis of W.

(a) If $[T]_\beta^\gamma$ is the identity matrix, show that T is an isomorphism.

(b) If T is an isomorphism, show β and γ can be chosen so that $[T]_\beta^\gamma$ is the identity matrix.

4. (12 points) Let V be a finite-dimensional inner product space with $T : V \to V$ linear.

(a) If $\langle T(\mathbf{v}), \mathbf{w} \rangle = 0$ for all $\mathbf{w}$ in V, show that $T(\mathbf{v}) = \mathbf{0}$.

(b) If T is one-to-one, show that T^* is onto.

(c) If T is onto, show that T^* is one-to-one.

5. (12 points) Suppose that $T : V \to V$ is a linear transformation on a finite-dimensional inner product space.

(a) Show that all eigenvalues of T^*T are nonnegative real numbers.

(b) If A is any square matrix with complex entries, show that the eigenvalues of A^*A are nonnegative real numbers.

(c) If B is any square matrix with complex entries, show that $\det(I + B^*B)$ is a positive real number.

6. (10 points) We say a matrix $B \in M_{n\times n}(\mathbb{C})$ is a "square root" of the matrix $A \in M_{n\times n}(\mathbb{C})$ if $B^2 = A$.

(a) If A is diagonalizable, show that it has a square root.

(b) Show that the non-diagonalizable matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ does not have a square root.

[Hint: If $B^2 = A$ explain why $B^4 = 0$, then consider the characteristic polynomial of B.]

7. (8 points) Suppose that A is an $n \times n$ matrix over the complex numbers of rank 1. Up to equivalence, what are the possible Jordan canonical forms of A?

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8. (8 points) Suppose $A \in M_{n\times n}(\mathbb{C})$. Prove that $\det(e^A) = e^{\mathrm{tr}(A)}$. [Hint: Put A in Jordan form.]

9. (12 points) Suppose V is an inner product space and $T : V \to V$ is linear and has an adjoint T^*.

(a) If T is Hermitian, show that $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$.

(b) If $T(\mathbf{v})$ is orthogonal to $\mathbf{v}$ for all $\mathbf{v} \in V$, show that T must be the zero transformation. [Hint: Apply the property to $\mathbf{v} = \mathbf{x} + \mathbf{y}$ and $\mathbf{w} = \mathbf{x} + i\mathbf{y}$, then take $\mathbf{y} = T\mathbf{x}$.]

(c) Suppose $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$. Show that T is Hermitian.

10. (16 points) Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$.

(a) Find the eigenvalues of A and a basis for each eigenspace.

(b) Find a formula for the nth power A^n.

(c) Solve the system of differential equations $y_1' = 2y_1 + 3y_2$, $y_2' = 3y_1 + 10y_2$.

(d) Find an invertible rational matrix Q such that $Q^T AQ$ is diagonal.

(e) Describe the shape of $2x^2+6xy+10y^2=1$ in $\mathbb{R}^2$ as one of the 3 standard conic sections.

(f) Classify the point (0,0) of $2x^2+6xy+10y^2$ as a local min, local max, or saddle point.

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11. (8 points) Let V be a finite-dimensional inner product space. Recall that we say $T : V \to V$ is an isometry when T^*T is the identity map on V. Prove that T is an isometry if and only if T is invertible and all of the singular values of T equal 1.