Math 4571: Advanced Linear Algebra

Practice Final A (Instructor: Dummit)

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 9 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	16	
2	8	
3	8	
4	12	
5	8	
6	10	
7	8	
8	8	
9	13	
10	9	
11	8	
12	12	
TOTAL	120	

1. (16 points) For each of the following, circle the correct response (there is no partial credit nor penalty for wrong answers).

True **False** The set $\{1 - x, 3 - x^2, 4 - x^3\}$ spans $P_3(\mathbb{R})$.

True **False** If $\dim(V) = 3$, then there is a linearly independent subset of V having exactly 3 elements.

True **False** If $T(\mathbf{v}) = T(\mathbf{w})$ implies $\mathbf{v} = \mathbf{w}$, then T is one-to-one.

True **False** If $S, T : V \to V$ then $[ST]_\alpha^\gamma = [S]_\alpha^\beta [T]_\beta^\gamma$ for any ordered bases α, β, γ of V.

True **False** In any inner product space, $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.

True **False** If S^* and T^* exist, then $(S + iT)^* = S^* + iT^*$.

True **False** A matrix is invertible if and only if it does not have 0 as an eigenvalue.

True **False** If A is any non-diagonalizable matrix having characteristic polynomial p, then $p(A)$ is the zero matrix.

True **False** Real symmetric matrices are diagonalizable.

True **False** Every bilinear form on $\mathbb{R}^n$ is diagonalizable.

True **False** The critical point $(0, 0, 0)$ of $f(x, y, z) = x^2 + y^2 - z^2$ is a saddle point.

True **False** If $A \in M_{n\times n}(\mathbb{C})$, the singular values of A are the eigenvalues of A^*A.

2. (8 points) An $n \times n$ matrix A is orthogonal when $A^T A = I_n$. If A is orthogonal, show that A^{-1} is also orthogonal.

3. (8 points) Suppose $T : V \to W$ is linear where V and W are finite-dimensional and $\dim V < \dim W$. Show that T is not onto.

4. (12 points) Suppose $T : V \to V$ is a linear transformation.

(a) If $\ker(T) = \ker(T^2)$, show that $\operatorname{im}(T) \cap \ker(T) = \{\mathbf{0}\}$.

(b) If $\operatorname{im}(T) = \operatorname{im}(T^2)$ and V is finite-dimensional, show that $\operatorname{im}(T) \cap \ker(T) = \{\mathbf{0}\}$.

(c) Show that the derivative map $D : \mathbb{R}[x] \to \mathbb{R}[x]$ has $\operatorname{im}(D) = \operatorname{im}(D^2)$, but also has $\operatorname{im}(D) \cap \ker(D)$ containing a nonzero polynomial.

5. (8 points) Prove $\sqrt{(a+c)^2+2(a+c)(b+d)+3(b+d)^2} \leq \sqrt{a^2+2ab+3b^2}+\sqrt{c^2+2cd+3d^2}$ for all real a, b, c, d.

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6. (10 points) Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a set of vectors in the real inner product space V, where $||\mathbf{u}_i|| = 1$ for each $1 \leq i \leq n$.

(a) If $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set, show that $||\mathbf{u}_i + \mathbf{u}_j|| = \sqrt{2}$ for every $1 \leq i, j \leq n$ with $i \neq j$.

(b) If $||\mathbf{u}_i + \mathbf{u}_j|| = \sqrt{2}$ for every $1 \leq i, j \leq n$ with $i \neq j$, show that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set.

7. (8 points) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $T : V \to V$ with respective eigenvalues λ_1 and λ_2, and $\mathbf{v}_1 + \mathbf{v}_2 \neq \mathbf{0}$. Show $\mathbf{v}_1 + \mathbf{v}_2$ is also an eigenvector of T if and only if $\lambda_1 = \lambda_2$.

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8. (8 points) Let $D : P_3(\mathbb{C}) \to P_3(\mathbb{C})$ be the derivative map. Show that D is not diagonalizable, and then find the Jordan canonical form of D.

9. (13 points) Suppose V is an inner product space and $T : V \to V$ is linear. Recall that we say T is an isometry if $\langle T(\mathbf{v}), T(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$; equivalently, when T^*T is the identity on V.

(a) If T is an isometry, show that any eigenvalue $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$.

(b) Conversely, suppose that V possesses an orthonormal basis of eigenvectors for T each of whose eigenvalues λ has $|\lambda| = 1$. Prove that T is an isometry. [Hint: Compute $\langle T(\mathbf{v}), T(\mathbf{w})\rangle$ in terms of this basis.]

(c) If T is an isometry that is also Hermitian, show that T^2 is the identity map.

10. (9 points) Let A, B be elements of $M_{n\times n}(F)$ for an arbitrary field F.

(a) If A and B are similar over F, prove that A^T and B^T are also similar over F.

(b) If A and B are congruent over F, prove that A^T and B^T are also congruent over F.

(c) If A and B are similar over F and invertible, show that A^{-1} and B^{-1} are similar over F.

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11. (8 points) Suppose that A is a symmetric positive-semidefinite matrix. Prove that the singular values of A are the eigenvalues of A.

12. (12 points) An $n \times n$ real matrix A is called a Gram matrix if there exists an $n \times n$ real matrix B with $A = B^T B$.

(a) Show that every Gram matrix is symmetric and positive semidefinite. [Hint: If $\mathbf{v}$ is an eigenvector, consider $\langle A\mathbf{v}, \mathbf{v} \rangle$ under the standard dot product.]

(b) Conversely, suppose that A is a positive-semidefinite symmetric matrix. Show that A is a Gram matrix.