E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Midterm 2 Solutions

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1. The first 8/12 correct were worth 1.5 each and the last 4/12 correct were worth 1 each.

   (a) $\boxed{\text{True}}$: we have $\langle 3\mathbf{x}, 2\mathbf{x}\rangle = 6\langle \mathbf{x}, \mathbf{x}\rangle \geq 0$ since $\langle \mathbf{x}, \mathbf{x}\rangle \geq 0$.

   (b) $\boxed{\text{True}}$: this is the triangle inequality.

   (c) $\boxed{\text{True}}$: an orthonormal basis can be constructed using Gram-Schmidt.

   (d) $\boxed{\text{True}}$: the vectors are pairwise orthogonal and nonzero, hence linearly independent hence a basis.

   (e) $\boxed{\text{True}}$: $||\mathbf{v}+\mathbf{w}||^2 = \langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$.

   (f) $\boxed{\text{False}}$: the orthogonal complement of the row space is the complex-conjugate of the nullspace.

   (g) $\boxed{\text{False}}$: taking adjoints reverses the order of composition, so $(ST)^* = T^*S^*$.

   (h) $\boxed{\text{True}}$: doing the multiplication shows $A\mathbf{v} = 7\mathbf{v}$ so $\mathbf{v}$ has eigenvalue 7.

   (i) $\boxed{\text{False}}$: eigenvalues are 1, 2, 3 hence each eigenspace has dimension 1, so it is diagonalizable.

   (j) $\boxed{\text{True}}$: by Cayley-Hamilton, $p(A) = 0$ so $A^3 + 2A - I_3 = 0$ so $A^3 + 2A = I_3$ as claimed.

   (k) $\boxed{\text{True}}$: diagonal matrices are in Jordan canonical form and the Jordan canonical form is unique.

   (l) $\boxed{\text{True}}$: if A and B are similar then the Jordan forms of A, B are also similar, but Jordan forms are similar iff they are equivalent.

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2. Part (a) was worth 6 points, while (c) and (d) were 3 each.

   (a) We verify the three parts of the definition. Let $\mathbf{v}_1 = (a_1, b_1)$, $\mathbf{v}_2 = (a_2, b_2)$, $\mathbf{v} = (a, b)$, $\mathbf{w} = (c, d)$.

   - [I1] $\langle \mathbf{v}_1 + r\mathbf{v}_2, \mathbf{w}\rangle = \langle (a_1 + ra_2, b_1 + rb_2), (c, d)\rangle = (a_1+ra_2)c + 2(a_1+ra_2)d + 2(b_1+rb_2)c + 5(b_1+rb_2)d = (a_1c + 2a_1d + 2b_1c + 5b_1d) + r(a_2c + 2a_2d + 2b_2c + 5b_2d) = \langle \mathbf{v}_1, \mathbf{w}\rangle + r\langle \mathbf{v}_2, \mathbf{w}\rangle$.
   - [I2] $\langle \mathbf{w}, \mathbf{v}\rangle = \langle (c, d), (a, b)\rangle = ca + 2cb + 2da + 5db = ac + 2ad + 2bc + 5bd = \langle \mathbf{v}, \mathbf{w}\rangle$.
   - [I3] $\langle \mathbf{v}, \mathbf{v}\rangle = a^2 + 4ab + 5b^2 = (a+2b)^2 + b^2$, which is always nonnegative. Furthermore, it is only zero when $a = b = 0$.

   (a) This is the Cauchy-Schwarz inequality applied to the inner product from (a): if $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$ then $\langle \mathbf{v}, \mathbf{w}\rangle^2 = (ac + 2ad + 2bc + 5bd)^2$, $||\mathbf{v}||^2 = a^2 + 4ab + 5b^2$, and $||\mathbf{w}||^2 = c^2 + 4cd + 5d^2$, so $\langle \mathbf{v}, \mathbf{w}\rangle^2 \leq \langle \mathbf{v}, \mathbf{v}\rangle \langle \mathbf{w}, \mathbf{w}\rangle$ reads $(ac + 2ad + 2bc + 5bd)^2 \leq (a^2 + 4ab + 5b^2)(c^2 + 4cd + 5d^2)$ as required.

   (b) This is the triangle inequality applied to the inner product from part (a): if $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$ then $\mathbf{v} + \mathbf{w} = (a + c, b + d)$, so $||\mathbf{v}|| = \sqrt{a^2 + 4ab + 5b^2}$, $||\mathbf{w}|| = \sqrt{c^2 + 4cd + 5d^2}$, and $||\mathbf{v} + \mathbf{w}|| = \sqrt{(a+c)^2 + 4(a+c)(b+d) + 5(b+d)^2}$. The inequality $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$ then reads $\sqrt{(a+c)^2 + 4(a+c)(b+d) + 5(b+d)^2} \leq \sqrt{a^2 + 4ab + 5b^2} + \sqrt{c^2 + 4cd + 5d^2}$ as required.

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3. We show both containments: $(W_1 + W_2)^\perp \subseteq W_1^\perp \cap W_2^\perp$ and $W_1^\perp \cap W_2^\perp \subseteq (W_1 + W_2)^\perp$.

   - If $\mathbf{w} \in (W_1 + W_2)^\perp$ then $\mathbf{w}$ is in particular orthogonal to every vector in W_1, hence is in $W_1^\perp$, and also is orthogonal to every vector in W_2, hence is in $W_2^\perp$. Then $\mathbf{w}$ is in $W_1^\perp \cap W_2^\perp$, as required.
   - Conversely, if $\mathbf{w} \in W_1^\perp \cap W_2^\perp$, then let $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$ be arbitrary, with $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$.
   - Then $\langle \mathbf{w}, \mathbf{w}_1\rangle = 0 = \langle \mathbf{w}, \mathbf{w}_2\rangle$ by the assumption that $\mathbf{w} \in W_1^\perp \cap W_2^\perp$, and so we have $\langle \mathbf{w}, \mathbf{w}_1 + \mathbf{w}_2\rangle = \langle \mathbf{w}, \mathbf{w}_1\rangle + \langle \mathbf{w}, \mathbf{w}_2\rangle = 0$.
   - This means $\mathbf{w}$ is orthogonal to $\mathbf{w}_1 + \mathbf{w}_2$, so since $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$ was arbitrary, we see $\mathbf{w} \in (W_1 + W_2)^\perp$. Thus, $W_1^\perp \cap W_2^\perp \subseteq (W_1 + W_2)^\perp$ as claimed.

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4. Suppose $\mathbf{v}$ is an eigenvector with eigenvalue λ, so that $T(\mathbf{v}) = \lambda\mathbf{v}$.
   - Then $\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, T^* T\mathbf{v} \rangle = \langle T\mathbf{v}, T\mathbf{v} \rangle = \langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle = \lambda\overline{\lambda} \langle \mathbf{v}, \mathbf{v} \rangle$.
   - Therefore, since $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ since $\mathbf{v}$ is nonzero (it is an eigenvector), we must have $\lambda\overline{\lambda} = 1$. But this is the same as $|\lambda|^2 = 1$, hence $|\lambda| = 1$ as claimed.

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5. Parts (a), (b), (c), (d) were each worth 2.5 points, (e) was worth 2 points, and (f) was worth 4 points.
   (a) The eigenvalues are the roots of the characteristic polynomial, hence they are $\boxed{1, 1, 1, 1, 2, 3}$ (so 1 has multiplicity 4 and the others have multiplicity 1).
   (b) The determinant is the product of the eigenvalues, which is $1^4 \cdot 2 \cdot 3 = \boxed{6}$ and the trace is the sum, which is $4 \cdot 1 + 2 + 3 = \boxed{9}$.
   (c) The eigenspace dimensions range from 1 to the multiplicity. So the 1-eigenspace can have dimension $\boxed{1, 2, 3, 4}$, the 2-eigenspace has dimension $\boxed{1}$, and the 3-eigenspace has dimension $\boxed{1}$.
   (d) By Cayley-Hamilton, the characteristic polynomial satisfies this requirement, so we can take $q(x) = \boxed{(x-1)^4(x-2)(x-3)}$.
   (e) A is diagonalizable if and only if the $\boxed{1}$-eigenspace has dimension $\boxed{4}$.
   (f) There must be one 1×1 block for each of $\lambda = 2$ and $\lambda = 3$, and then the possible Jordan block sizes for $\lambda = 1$ are 4, 3-1, 2-2, 2-1-1, and 1-1-1-1, yielding these five possibilities (only 3 are required):

$$\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & & & & \\ & 1 & & & & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 2 & \\ & & & & & 3 \end{bmatrix}.$$

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6. Each part was worth 4 points.
   (a) Suppose $P^{-1}AP = D$ is diagonal. Then $D^k = (P^{-1}AP)^k = P^{-1}A^kP = P^{-1}0P = 0$ is the zero matrix, so since D is diagonal, it must itself be the zero matrix. Then $A = PDP^{-1} = 0$ is also zero.
   (b) By the spectral mapping theorem, the eigenvalues of A^k are the kth powers of the eigenvalues of A. So since A^k is the zero matrix, which only has eigenvalues of 0, the kth powers of the eigenvalues of A are all 0, meaning the eigenvalues themselves are zero.
   (c) Since $\mathbb{C}$ is algebraically closed, the characteristic polynomial of A splits completely. Thus, if the only eigenvalues of A are $\lambda = 0$, then the characteristic polynomial must be $p(t) = (t - 0)^n$. Then by Cayley-Hamilton, we have $p(A) = A^n = 0$, so A is nilpotent.
   (d) By (b) and (c), being nilpotent is equivalent to having all eigenvalues equal to 0. The possible Jordan block sizes are 3, 2-1, and 1-1-1, yielding $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
