

1. Answer the following true/false questions, where $S, T : V \rightarrow W$ are linear transformations.
 - (a) The vector space \mathbb{R}^4 has exactly one inner product.
 - (b) The Cauchy-Schwarz inequality is true in \mathbb{R}^n but false in any other space.
 - (c) In any inner product space, $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2 \rangle = \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_2, \mathbf{w}_2 \rangle$.
 - (d) In any inner product space, $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.
 - (e) In any inner product space, $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|$.
 - (f) Every finite-dimensional inner product space has an orthonormal basis.
 - (g) The set $\frac{1}{3}(1, 2, 2), \frac{1}{\sqrt{5}}(0, 1, -1), \frac{1}{\sqrt{18}}(-4, 1, 1)$ is an orthonormal basis for \mathbb{R}^3 , under the dot product.
 - (h) Every linear transformation has an adjoint.
 - (i) If S^* and T^* exist, then $(S + iT)^* = S^* + iT^*$.
 - (j) The vector $\mathbf{v} = e^{-x} - 2e^x$ is an eigenvector for the map $D^2 : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ given by $D^2(f) = f''$.
 - (k) If two matrices have the same eigenvalues, then they are similar.
 - (l) If two matrices are similar, then they have the same eigenvalues.
 - (m) If two matrices are similar, then they have the same eigenvectors.
 - (n) A matrix is invertible if and only if it does not have 0 as an eigenvalue.
 - (o) If the characteristic polynomial for A is $(t - 1)^4(t - 2)^2$, then the 1-eigenspace of A is 4-dimensional.
 - (p) If the characteristic polynomial for A is $(t - 1)^4(t - 2)^2$, then the determinant of A is 2.
 - (q) Any $n \times n$ matrix having only one eigenvalue is diagonalizable.
 - (r) Any $n \times n$ matrix having only one eigenvalue is never diagonalizable.
 - (s) The only $n \times n$ matrix A with $(A - I_n)^n = \mathbf{0}$ is the identity matrix.
 - (t) If A is any non-diagonalizable matrix having characteristic polynomial p , then $p(A)$ is the zero matrix.
 - (u) If J_1 and J_2 are two matrices in Jordan canonical form and J_1 is similar to J_2 , then $J_1 = J_2$.
 - (v) If two matrices have the same Jordan canonical form, then they are similar.
 - (w) If two matrices are similar, then they have the same Jordan canonical form.
 - (x) Real symmetric matrices are diagonalizable.

- 2. For each of the following pairings $\langle \cdot, \cdot \rangle$ on the vector space V , determine whether or not it is an inner product on V :
 - (a) $V = C[0, 1]$, with $\langle f, g \rangle = f'(0)g(1)$.
 - (b) $V = \mathbb{R}^2$, with $\langle (a, b), (c, d) \rangle = 4ac + ad + bc + 4bd$.
 - (c) $V = P_3(\mathbb{C})$, with $\langle p, q \rangle = \int_0^1 p(x)\overline{q(x)} dx$.

- 3. Prove that $\sqrt{\frac{a+b}{a+b+c}} + \sqrt{\frac{b+c}{a+b+c}} + \sqrt{\frac{c+a}{a+b+c}} \leq \sqrt{6}$ for any positive real numbers a, b, c .

- 4. Prove $\sqrt{a^2 + 2ab + 3b^2} + \sqrt{c^2 + 2cd + 3d^2} \leq \sqrt{(a+c)^2 + 2(a+c)(b+d) + 3(b+d)^2}$ for all real a, b, c, d .

- 5. Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a set of vectors in the real inner product space V , where $\|\mathbf{u}_i\| = 1$ for each $1 \leq i \leq n$.
 - (a) If $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set, show that $\|\mathbf{u}_i + \mathbf{u}_j\| = \sqrt{2}$ for every $1 \leq i, j \leq n$ with $i \neq j$.
 - (b) If $\|\mathbf{u}_i + \mathbf{u}_j\| = \sqrt{2}$ for every $1 \leq i, j \leq n$ with $i \neq j$, show that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set.

- 6. Suppose V is a finite-dimensional inner product space and $T : V \rightarrow V$ is linear.
 - (a) If $\langle T(\mathbf{v}), \mathbf{w} \rangle = 0$ for all \mathbf{w} in V , show that $T(\mathbf{v}) = \mathbf{0}$.
 - (b) If T is one-to-one, show that T^* is onto.
 - (c) If T is onto, show that T^* is one-to-one.

7. Suppose V is an inner product space with subspaces W_1 and W_2 . Show $W_1^\perp + W_2^\perp \subseteq (W_1 \cap W_2)^\perp$.

8. Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \rightarrow V$ is a linear transformation possessing an adjoint T^* . Recall that $S : V \rightarrow V$ is Hermitian when $S = S^*$.

- (a) Show that $T + T^*$, $i(T - T^*)$, T^*T , and TT^* are all Hermitian.
 - (b) Prove that $\ker(T^*T) = \ker(T)$.
 - (c) Suppose T is Hermitian. Show that $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is a real number for any vector \mathbf{v} .
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9. Suppose that $T : V \rightarrow V$ is a linear transformation on a finite-dimensional inner product space.

- (a) Show that all eigenvalues of T^*T are nonnegative real numbers.
 - (b) If A is any square matrix with complex entries, show that the eigenvalues of A^*A are nonnegative real numbers.
 - (c) If B is any square matrix with complex entries, show that $\det(I + B^*B)$ is a positive real number.
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10. For each matrix A , find the Jordan canonical form of A .

(a) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. (c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. (d) $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

11. Suppose \mathbf{v}_1 and \mathbf{v}_2 are eigenvectors of $T : V \rightarrow V$ with respective eigenvalues λ_1 and λ_2 , and $\mathbf{v}_1 + \mathbf{v}_2 \neq \mathbf{0}$. Show that $\mathbf{v}_1 + \mathbf{v}_2$ is also an eigenvector of T if and only if $\lambda_1 = \lambda_2$.

12. We say a matrix $B \in M_{n \times n}(\mathbb{C})$ is a “square root” of the matrix $A \in M_{n \times n}(\mathbb{C})$ if $B^2 = A$.

- (a) If A is diagonalizable, show that it has a square root.
 - (b) Show that the non-diagonalizable matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ does not have a square root. [Hint: If $B^2 = A$ explain why $B^4 = 0$ and then consider the characteristic polynomial of B .]
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13. Suppose V is a real vector space and that $T : V \rightarrow V$ has the property that $T^2 = I$.

- (a) Show that the only possible eigenvalues of T are 1 and -1 .
 - (b) Suppose that $(T - I)^2\mathbf{v} = \mathbf{0}$. Show in fact that $(T - I)\mathbf{v} = \mathbf{0}$.
 - (c) Suppose that $(T + I)^2\mathbf{v} = \mathbf{0}$. Show in fact that $(T + I)\mathbf{v} = \mathbf{0}$.
 - (d) Show that any generalized eigenvector of T is actually an eigenvector of T .
 - (e) If V is finite-dimensional, show that T is diagonalizable, and describe its diagonalization.
 - (f) Let $T : M_{3 \times 3}(\mathbb{R}) \rightarrow M_{3 \times 3}(\mathbb{R})$ be the transpose map. Show that T is diagonalizable and find its diagonalization.
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14. Suppose that A is an $n \times n$ matrix over the complex numbers of rank 1.

- (a) Up to equivalence, what are the possible Jordan canonical forms of A ?
 - (b) For each matrix you listed in part (a), what is its characteristic polynomial?
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15. Let $D : P_3(\mathbb{C}) \rightarrow P_3(\mathbb{C})$ be the derivative map. Show that D is not diagonalizable, and then find the Jordan canonical form of D .
