1. (a) False: there are many.
   (b) False: Cauchy-Schwarz is always true.
   (c) False: the cross-terms are missing.
   (d) False: should be $\overline{\langle \mathbf{v}, \mathbf{w} \rangle} = \langle \mathbf{w}, \mathbf{v} \rangle$.
   (e) True: this is Cauchy-Schwarz.
   (f) True: use Gram-Schmidt to construct.
   (g) False: middle vector does not have length 1.
   (h) False: HW7 has an example of T with no T^*.
   (i) False: $(S + iT)^* = S^* - iT^*$.
   (j) True: $D^2(\mathbf{v}) = \mathbf{v}$ as one can check.
   (k) False: a 2×2 Jordan block and λI_2 are not similar.
   (l) True: similar matrices have the same char. poly.
   (m) False: similarity doesn't preserve eigenvectors.
   (n) True: $\ker(A) = 0$ iff the 0-eigenspace is 0.
   (o) False: the 1-eigenspace is *at most* 4-dimensional.
   (p) False: the determinant is $1^4 2^2 = 4$.
   (q) False: take a Jordan block of size at least 2.
   (r) False: the identity matrix is diagonalizable.
   (s) False: take a Jordan matrix with all eigenvalues 1.
   (t) True: Cayley-Hamilton applies to all matrices.
   (u) False: the Jordan blocks could be rearranged.
   (v) True: change bases from $A \to J \to B$.
   (w) True: the Jordan form is invariant under similarity.
   (x) True: this follows from the spectral theorem.

2. (a) No: although [I1] and [I2] are satisfied, [I3] fails since $\langle f, f \rangle = f'(0)f(1)$ can be negative.
   (b) Yes: [I1] and [I2] are direct algebra checks, and for [I3], $\langle (a,b), (a,b) \rangle = 4a^2 + 2ab + 4b^2 = (2a + b/2)^2 + (17/4)b^2 \geq 0$ and equality holds only for $a = b = 0$.
   (c) Yes: [I1] and [I2] are direct algebra checks, and for [I3], $\langle p, p \rangle = \int_0^1 |p(x)|^2 \, dx \geq 0$ since the integrand is nonnegative, and the integral equals zero only when $|p(x)| = 0$ on $[0,1]$, which is to say $p(x)$ is zero.

3. Take $\mathbf{v} = \left(\sqrt{\frac{a+b}{a+b+c}}, \sqrt{\frac{b+c}{a+b+c}}, \sqrt{\frac{c+a}{a+b+c}}\right)$ and $\mathbf{w} = (1,1,1)$: as $||\mathbf{v}|| = \sqrt{2}$ and $||\mathbf{w}|| = \sqrt{3}$, Cauchy-Schwarz gives $\mathbf{v} \cdot \mathbf{w} \leq ||\mathbf{v}|| \, ||\mathbf{w}||$ and this is precisely the given inequality.

4. Observe that the pairing $\langle (a,b), (c,d) \rangle = ac + ad + bc + 3bd$ is an inner product on $\mathbb{R}^2$: [I1] and [I2] are direct calculations and for [I3] we have $\langle (a,b), (a,b) \rangle = a^2 + 2ab + 3b^2 = (a+b)^2 + 2b^2 \geq 0$ with equality only for $a = b = 0$. The desired inequality is just the triangle inequality for $\mathbf{v} = (a,b)$ and $\mathbf{w} = (c,d)$: $||\mathbf{v}|| + ||\mathbf{w}|| \leq ||\mathbf{v} + \mathbf{w}||$.

5. (a) $||\mathbf{u}_i + \mathbf{u}_j||^2 = \langle \mathbf{u}_i + \mathbf{u}_j, \mathbf{u}_i + \mathbf{u}_j \rangle = \langle \mathbf{u}_i, \mathbf{u}_i \rangle + \langle \mathbf{u}_i, \mathbf{u}_j \rangle + \langle \mathbf{u}_j, \mathbf{u}_i \rangle + \langle \mathbf{u}_j, \mathbf{u}_j \rangle = 1 + 0 + 0 + 1 = 2$.
   (b) $2 = ||\mathbf{u}_i + \mathbf{u}_j||^2 = \langle \mathbf{u}_i, \mathbf{u}_i \rangle + \langle \mathbf{u}_i, \mathbf{u}_j \rangle + \langle \mathbf{u}_j, \mathbf{u}_i \rangle + \langle \mathbf{u}_j, \mathbf{u}_j \rangle = 2 + 2\langle \mathbf{u}_i, \mathbf{u}_j \rangle$ so $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$.

6. (a) Set $\mathbf{w} = T(\mathbf{v})$: then $\langle T(\mathbf{v}), T(\mathbf{v}) \rangle = 0$ hence $T(\mathbf{v}) = 0$.
   (b) As proven in class, since V is finite-dimensional, $\ker(T)$ and $\operatorname{im}(T^*)$ are orthogonal complements. If T is one-to-one then $\ker(T) = 0$ so $\operatorname{im}(T^*) = \{\mathbf{0}\}^\perp = V$ hence T^* is onto.
   (c) As proven in class, $\ker(T^*)$ and $\operatorname{im}(T)$ are orthogonal complements. If T is onto then $\operatorname{im}(T) = V$ so $[\operatorname{im}(T)]^\perp = \{\mathbf{0}\}$ hence $\ker(T^*) = \{\mathbf{0}\}$ so T^* is one-to-one.

7. Suppose $\mathbf{w} \in W_1^\perp + W_2^\perp$: then $\mathbf{w} = \mathbf{w}_1^\perp + \mathbf{w}_2^\perp$ for $\mathbf{w}_1^\perp \in W_1^\perp$ and $\mathbf{w}_2^\perp \in W_2^\perp$. Now for any $\mathbf{v} \in W_1 \cap W_2$, we have $\langle \mathbf{v}, \mathbf{w}_1 \rangle = 0$ since $\mathbf{v} \in W_1$ and $\mathbf{w}_1^\perp \in W_1^\perp$; also $\langle \mathbf{v}, \mathbf{w}_2 \rangle = 0$ since $\mathbf{v} \in W_2$ and $\mathbf{w}_2 \in W_2^\perp$. Then $\langle \mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2 \rangle = \langle \mathbf{v}, \mathbf{w}_1 \rangle + \langle \mathbf{v}, \mathbf{w}_2 \rangle = 0$ hence $\mathbf{w} \in (W_1 \cap W_2)^\perp$.

8. (a) First we have $(T + T^*)^* = T^* + T^{**} = T + T^*$. Second we have $[iT - iT^*]^* = -iT^* + iT^{**} = iT - iT^*$. Third we have $(T^*T)^* = T^*T^{**} = T^*T$. Finally we have $(TT^*)^* = T^{**}T^* = TT^*$.
   (b) If $T(\mathbf{v}) = \mathbf{0}$ then $T^*T(\mathbf{v}) = \mathbf{0}$ so $\ker(T) \subseteq \ker(T^*T)$. Conversely if $T^*T(\mathbf{v}) = \mathbf{0}$ then $0 = \langle \mathbf{v}, T^*T(\mathbf{v}) \rangle = \langle T(\mathbf{v}), T(\mathbf{v}) \rangle$ so $T(\mathbf{v}) = 0$.
   (c) We have $\langle T(\mathbf{v}), \mathbf{v} \rangle = \overline{\langle \mathbf{v}, T(\mathbf{v}) \rangle} = \overline{\langle \mathbf{v}, T^*(\mathbf{v}) \rangle} = \overline{\langle T(\mathbf{v}), \mathbf{v} \rangle}$ so $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is real.

9. (a) Suppose $T^*T(\mathbf{v}) = \lambda\mathbf{v}$. Then $||T(\mathbf{v})||^2 = \langle T(\mathbf{v}), T(\mathbf{v})\rangle = \langle \mathbf{v}, T^*T(\mathbf{v})\rangle = \langle \mathbf{v}, \lambda\mathbf{v}\rangle = \overline{\lambda}\,\langle \mathbf{v}, \mathbf{v}\rangle = \overline{\lambda}\,||\mathbf{v}||^2$. Therefore we see $\overline{\lambda} = ||T(\mathbf{v})||^2 / ||\mathbf{v}||^2 \geq 0$ and thus λ is a nonnegative real number.
   (b) Applying (a) to the linear transformation $T(\mathbf{v}) = A\mathbf{v}$ yields the result immediately.
   (c) The eigenvalues of $I + B^*B$ are 1 plus the eigenvalues of B^*B hence they are all positive. Then $\det(I + B^*B)$ is the product of the eigenvalues (with multiplicity) so it is also positive.

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10. (a) $\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. These can be found by computing eigenspace dimensions and identifying the number of Jordan blocks and their possible sizes.

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11. If $\lambda_1 = \lambda_2$ then $\mathbf{v}_1 + \mathbf{v}_2$ is in the λ_1-eigenspace. Conversely, since $T(\mathbf{v}_1 + \mathbf{v}_2) = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$ if $T(\mathbf{v}_1 + \mathbf{v}_2) = \mu(\mathbf{v}_1 + \mathbf{v}_2)$ then $(\lambda_1 - \mu)\mathbf{v}_1 + (\lambda_2 - \mu)\mathbf{v}_2 = \mathbf{0}$. If $\mu = \lambda_1 = \lambda_2$ then this relation holds trivially; otherwise at least one coefficient is nonzero so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, and this can only occur when $\lambda_1 = \lambda_2$, so either way $\lambda_1 = \lambda_2$.

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12. (a) Clearly any diagonal matrix D has a square root E (just take square roots of each of its diagonal entries). Then if A is diagonalizable with $Q^{-1}AQ = D$, for $B = QEQ^{-1}$ we have $B^2 = QE^2Q^{-1} = QDQ^{-1} = A$.
    (b) Suppose $B^2 = A$. Then since $A^2 = 0$ we see $B^4 = 0$. If $\mathbf{v}$ is an eigenvector of B with eigenvalue λ then $\mathbf{0} = 0\mathbf{v} = B^4\mathbf{v} = \lambda^4\mathbf{v}$ hence $\lambda = 0$. Thus, the only eigenvalues of B are zero, so the characteristic polynomial of B is $p(t) = t^2$. By Cayley-Hamilton this implies $B^2 = 0$, but $B^2 = A$ which would give $A = 0$, contradiction.

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13. (a) If $T(\mathbf{v}) = \lambda\mathbf{v}$ then $\mathbf{v} = T^2(\mathbf{v}) = T(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$ so $\lambda^2 = 1$ so $\lambda = \pm 1$.
    (b) If $(T - I)^2\mathbf{v} = \mathbf{0}$ then $(T^2 - 2T + I)\mathbf{v} = \mathbf{0}$ but $T^2 = I$ so this yields $(-2T + 2)\mathbf{v} = \mathbf{0}$ hence $(T - I)\mathbf{v} = \mathbf{0}$.
    (c) Like (b), expanding $(T + I)^2\mathbf{v} = \mathbf{0}$ yields $(2T + 2)\mathbf{v} = \mathbf{0}$ so $(T + I)\mathbf{v} = \mathbf{0}$.
    (d) The only eigenvalues are ± 1 so a generalized eigenvector satisfies $(T - I)^d\mathbf{v} = \mathbf{0}$ or $(T + I)^d\mathbf{v} = \mathbf{0}$. In the first case taking $\mathbf{w} = (T - I)^{d-2}\mathbf{v}$ gives $(T - I)^2\mathbf{w} = \mathbf{0}$ so (b) gives $(T - I)\mathbf{w} = \mathbf{0}$ so $(T - I)^{d-1}\mathbf{v} = \mathbf{0}$. By induction this reduces to $(T - I)\mathbf{v} = \mathbf{0}$; similarly $(T + I)^d\mathbf{v} = \mathbf{0}$ reduces to $(T + I)\mathbf{v} = \mathbf{0}$: thus all generalized eigenvectors are eigenvectors.
    (e) Since the eigenvalues are $\pm 1 \in \mathbb{R}$, V has a basis of generalized eigenvectors, hence by (d) V has a basis of eigenvectors. Thus T is diagonalizable, and its diagonalization has entries ± 1.
    (f) Note $T^2 = I$ so by (e) T is diagonalizable. It is not hard to see the 1-eigenspace (symmetric matrices) has dimension 6 while the (-1)-eigenspace (skew-symmetric matrices) has dimension 3, so the diagonalization has 6 diagonal entries of $+1$ and 3 of -1.

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14. (a) Rank is preserved by similarity, so the Jordan form must also have rank 1. There are two possibilities: either $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (for any $\lambda \neq 0$) or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
    (b) The characteristic polynomials are $x^2(x - \lambda)$ and x^3 respectively.

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15. With $\beta = \{1, x, x^2, x^3\}$ we see $[D]_\beta^\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is upper-triangular, so the eigenvalues are $\lambda = 0, 0, 0, 0$ and the 0-eigenspace is 1-dimensional, so D is not diagonalizable. Its Jordan form must have one block with eigenvalue 0, so the Jordan form is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
