Math 4571: Advanced Linear Algebra

Midterm 2 (Instructor: Dummit)

April 1st, 2026

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 6 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	16	
2	12	
3	10	
4	10	
5	16	
6	16	
TOTAL	80	

1. (16 points) For each of the following, circle the correct response (there is no partial credit nor penalty for wrong answers).

True **False** For any vector $\mathbf{x}$ in an inner product space, $\langle 3\mathbf{x}, 2\mathbf{x} \rangle \geq 0$ is always true.

True **False** In any inner product space, $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$ for any vectors $\mathbf{v}$, $\mathbf{w}$.

True **False** Every finite-dimensional inner product space has an orthonormal basis.

True **False** The vectors $(2, 1, -1)$, $(0, 1, 1)$, $(1, -1, 1)$ are an orthogonal basis of $\mathbb{R}^3$.

True **False** If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then $||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$.

True **False** The orthogonal complement of a matrix's row space is its column space.

True **False** If $S, T : V \to V$ have adjoints, so does ST and $(ST)^* = S^*T^*$.

True **False** The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for the matrix $\begin{bmatrix} 2 & 1 & 4 \\ 2 & 2 & 3 \\ 6 & 0 & 1 \end{bmatrix}$.

True **False** The matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable.

True **False** If A is a 3×3 matrix and $\det(tI_3 - A) = t^3 + 2t - 1$, then $A^3 + 2A = I_3$.

True **False** A is diagonalizable if and only if its Jordan canonical form is diagonal.

True **False** If A and B are similar matrices, their Jordan canonical forms are equivalent.

2. (12 points) Let $V = \mathbb{R}^2$.

(a) Show that the pairing $\langle (a,b),(c,d)\rangle = ac + 2ad + 2bc + 5bd$ is an inner product on V.

(b) Show that $(ac + 2ad + 2bc + 5bd)^2 \le (a^2 + 4ab + 5b^2)(c^2 + 4cd + 5d^2)$ for any real a, b, c, d.

(c) Show that $\sqrt{(a+c)^2 + 4(a+c)(b+d) + 5(b+d)^2} \le \sqrt{a^2 + 4ab + 5b^2} + \sqrt{c^2 + 4cd + 5d^2}$ for any real a, b, c, d.

3. (10 points) Let V be a finite-dimensional inner product space with subspaces W_1 and W_2. Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

4. (10 points) Suppose V is an inner product space and $T : V \to V$ possesses an adjoint T^* such that T^*T is the identity map on V. If λ is an eigenvalue of T, show that $|\lambda| = 1$.

5. (16 points) Suppose A is a 6×6 complex matrix with characteristic polynomial

$$p(t) = (t-1)^4(t-2)(t-3).$$

(a) Find the eigenvalues of A and their multiplicities.

(b) Find the determinant and trace of A.

(c) Find all possible dimensions of EACH of the eigenspaces of A.

(d) Find a nonzero polynomial $q(x)$ such that $q(A)$ is guaranteed to be the zero matrix.

(e) Fill in the blanks:

A will be diagonalizable if and only if the ______-eigenspace has dimension ______.

(f) Find three (inequivalent) possible Jordan canonical forms for A.

6. (16 points) A matrix $A \in M_{n\times n}(\mathbb{C})$ is called nilpotent if A^k is the zero matrix for some positive integer k.

(a) If A is diagonalizable and nilpotent, show that it is the zero matrix.

(b) If A is nilpotent, show that the only eigenvalue of A is $\lambda = 0$.

(c) Conversely, if the only eigenvalue of A is $\lambda = 0$, show that A is nilpotent. [Hint: Consider the characteristic polynomial.]

(d) Find all possible Jordan canonical forms for a 3×3 nilpotent matrix.