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1. The first 8/12 correct were worth 1.5 each and the last 4/12 correct were worth 1 each.

   (a) **False**: in fact any such matrix must have determinant zero, so it is NOT invertible.

   (b) **True**: if $a(1+x)+b(2+x^2)+c(3+x^3)=0$ then $a=b=c=0$.

   (c) **False**: since $M_{2\times 2}(\mathbb{R})$ is 4-dimensional, any spanning set must have at least 4 matrices.

   (d) **True**: any basis for W is linearly independent hence can be extended to a basis for V; this basis of V contains a basis for W.

   (e) **False**: in general $\{T(\mathbf{v}_1),\dots,T(\mathbf{v}_n)\}$ spans $\text{im}(T)$ but it need not be linearly independent, e.g., if T is the zero transformation.

   (f) **False**: the matrix is in row-echelon form with 3 pivots so its nullspace has dimension $5-3=2$, not 3.

   (g) **True**: this map is an isomorphism since it has an inverse $T^{-1}(A)=A^T/2$.

   (h) **True**: necessarily V must be infinite-dimensional, but we have seen several examples of such T, such as the derivative on polynomials and the left-shift operator on sequences.

   (i) **True**: this is a theorem we proved in class.

   (j) **True**: a linear transformation is characterized by its values on a basis, so since S and T are equal on a basis, they are identically equal.

   (k) **False**: this is in general the change-of-basis matrix from α to β, which is only the identity when $\alpha=\beta$.

   (l) **True**: the bases and transformations are composed correctly here.

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2. Each part was worth 4 points.

   (a) Evaluating S on the four matrices in β shows that the matrix is $\boxed{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$.

   (b) We induct on n. For the base case $n=1$ we have $S^2(A)=S(A+A^T)=(A^T+A)+(A+A^T)=2(A+A^T)=2S(A)$ as required. For the inductive step, suppose $S^{n+1}=2^nS$. Then $S^{n+2}(A)=S^{n+1}(S(A))=2^nS(S(A))=2^{n+1}S(A)$ using the previous calculation.
   Alternatively, we could verify by induction that $M^{n+1}=2^nM$ for the matrix calculated in (a).

   (c) The kernel of S is all matrices A with $A+A^T=0$ so that $A^T=-A$ (i.e., skew-symmetric matrices): solving $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ yields $a=d=0$ and $b=-c$, yielding a basis $\boxed{\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}}$ for $\ker(T)$.
   For the image, note that $A^T+A=\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = 2a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b+c)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, yielding a basis $\boxed{\left\{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right\}}$ for $\text{im}(T)$.

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3. Each part was worth 4 points.

   (a) Suppose A is orthogonal. Taking the determinant of $A^T=A^{-1}$ and using $\det(A^T)=\det(A)$ and $\det(A^{-1})=1/\det(A)$ yields $\det(A)=1/\det(A)$ so $\det(A)^2=1$ so $\det(A)=1$ or -1.

   (b) Suppose A,B are orthogonal so $A^T=A^{-1}$ and $B^T=B^{-1}$. Then $(AB)^T=B^TA^T=B^{-1}A^{-1}=(AB)^{-1}$ so AB is also orthogonal.

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4. We verify the subspace criterion.

   - [S1]: Since $T(\mathbf{0}_V) = \mathbf{0}_W$ and $\mathbf{0}_W \in W_1$ since W_1 is a subspace, we have $\mathbf{0}_V \in T^{-1}(W_1)$.
   - [S2]: If $\mathbf{v}_1, \mathbf{v}_2 \in T^{-1}(W_1)$ then $T(\mathbf{v}_1), T(\mathbf{v}_2)$ are in W_1. But then so is $T(\mathbf{v}_1+\mathbf{v}_2) = T(\mathbf{v}_1)+T(\mathbf{v}_2)$ since W_1 is a subspace, and thus $\mathbf{v}_1 + \mathbf{v}_2 \in T^{-1}(W_1)$.
   - [S3]: If $\mathbf{v} \in T^{-1}(W_1)$ then $T(\mathbf{v})$ is in W_1 hence so is $\alpha T(\mathbf{v}) = T(\alpha\mathbf{v})$ since W_1 is a subspace, and thus $\alpha\mathbf{v} \in T^{-1}(W_1)$.

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5. Suppose $\mathbf{v} \in \ker(T)$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis we have $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ for some a_1, a_2, a_3. Then $\mathbf{0} = T(\mathbf{v}) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + a_3T(\mathbf{v}_3)$ but since $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent this means $a_1 = a_2 = a_3 = 0$ hence $\mathbf{v} = \mathbf{0}$.

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6. By the nullity-rank theorem, $\dim(\ker T) + \dim(\operatorname{im} T) = \dim(V) = 2026$. Furthermore, since $\operatorname{im} T$ is a subspace of W, we have $\dim(\operatorname{im} T) \leq 1900$. Thus, $\dim(\ker T) \geq 2026 - 1900 = 126$. Now simply take $\mathbf{v}_1, \ldots, \mathbf{v}_{100}$ to be the first 100 vectors in a basis for $\ker(T)$: then these vectors are by definition linearly independent and have $T(\mathbf{v}_i) = \mathbf{0}$ for each i.

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7. Solution 1: If $\mathbf{v}$ is in $\ker(T)$ so that $T(\mathbf{v}) = \mathbf{0}$, then $\mathbf{v} = T^3(\mathbf{v}) = T^2(T(\mathbf{v})) = T^2(\mathbf{0}) = \mathbf{0}$, so T is one-to-one. Also, for any vector $\mathbf{w}$, if $\mathbf{v} = T^2(\mathbf{w})$ then $T(\mathbf{v}) = T^3(\mathbf{w}) = \mathbf{w}$, so T is onto. Hence, T is an isomorphism.

   Solution 2: If $T^3 = I$ then $T(T^2) = (T^2)T = I$, so T^2 is a two-sided inverse for T. The only transformations with inverses are isomorphisms, so T must be an isomorphism.

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8. Each part was worth 3 points.

   (a) Suppose $\mathbf{w}$ is in $\operatorname{im}(T)$. Then there exists $\mathbf{v}$ with $\mathbf{w} = T(\mathbf{v})$. Then $T(\mathbf{w}) = T(T(\mathbf{v})) = \mathbf{0}$, meaning that $\mathbf{w}$ is in $\ker(T)$. Thus, $\operatorname{im}(T)$ is contained in $\ker(T)$. [Note: This part is a special case of problem 5 on homework 4, and citing that problem was also acceptable.]

   (b) By part (a), $\dim(\operatorname{im}T) \leq \dim(\ker T)$, and by nullity-rank, $\dim(\operatorname{im}T) + \dim(\ker T) = 2$. Thus, $\dim(\operatorname{im}T)$ is either 0 or 1. But the dimension of the image cannot be zero, since this would imply that T is the zero transformation. Thus, $\dim(\operatorname{im}T) = 1$.

   (c) Since $\{\mathbf{v}, \mathbf{w}\}$ has size $2 = \dim(\mathbb{R}^2)$ it is enough to show that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. But if $\mathbf{0} = a\mathbf{v} + b\mathbf{w}$ then $\mathbf{0} = T(\mathbf{0}) = T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w}) = b\mathbf{v}$, so since $\mathbf{v}$ is nonzero, $b = 0$. Then $a\mathbf{v} = \mathbf{0}$ so $a = 0$. So $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, hence a basis.

   (d) Since $T(\mathbf{v}) = \mathbf{0} = 0\mathbf{v} + 0\mathbf{w}$ and $T(\mathbf{w}) = \mathbf{v} = 1\mathbf{v} + 0\mathbf{w}$, the matrix is $[T]_\beta^\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ as claimed.
