1. Answer the following true/false questions, where $T : V \to W$ is a linear transformation.
   (a) The vectors $\langle 1,1\rangle$, $\langle 4,1\rangle$, $\langle 2,3\rangle$ span $\mathbb{R}^2$.
   (b) The vectors $\langle 1,1\rangle$, $\langle 4,1\rangle$, $\langle 2,3\rangle$ are linearly independent in $\mathbb{R}^2$.
   (c) The set $\{1-x,\ 3-x^2,\ 4-x^3\}$ spans $P_3(\mathbb{R})$.
   (d) The set $\{1-x,\ 3-x^2,\ 4-x^3\}$ is linearly independent in $P_3(\mathbb{R})$.
   (e) If W is a subspace of V, then $\dim(W) \le \dim(V)$.
   (f) Every finite-dimensional vector space has a finite basis.
   (g) The set of vectors in any vector space V forms a basis for V.
   (h) If $\dim(V) = 3$, then every set of at least 3 vectors spans V.
   (i) If $\dim(V) = 3$, then there is a linearly independent subset of V having exactly 3 elements.
   (j) If $\dim(V) = 3$, then no linearly independent subset of V can have exactly 2 elements.
   (k) The dimension of a vector space is always positive.
   (l) The vector $\mathbf{v} = e^{-x} - 2e^x$ is in the kernel of the map $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ with $T(f) = f'' - f$.
   (m) The vector $\mathbf{v} = \langle 2,3,4\rangle$ is in the image of the map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(a,b) = \langle a-b, a+b, a-b\rangle$.
   (n) The vector $\mathbf{v} = x^2$ is in the image of the map $T : \mathbb{R}[x] \to \mathbb{R}[x]$ with $T(f) = \int_0^x f(t)\,dt$.
   (o) If $T(\mathbf{v}) = T(\mathbf{w})$ implies $\mathbf{v} = \mathbf{w}$, then T is one-to-one.
   (p) If for any $\mathbf{v}$ in V there exists $\mathbf{w}$ in W with $T(\mathbf{v}) = \mathbf{w}$, then T is onto.
   (q) If for any $\mathbf{w}$ in W there is a unique $\mathbf{v}$ in V with $T(\mathbf{v}) = \mathbf{w}$, then T is an isomorphism.
   (r) There exists a linear $T : \mathbb{R}^5 \to \mathbb{R}^4$ whose nullity is 2 and whose rank is 2.
   (s) There exists a linear $T : \mathbb{R}^5 \to \mathbb{R}^4$ whose nullity is 4 and whose rank is 1.
   (t) There exists a linear $T : \mathbb{R}^5 \to \mathbb{R}^4$ that is onto but not one-to-one.
   (u) There exists a linear $T : \mathbb{R}^5 \to \mathbb{R}^4$ that is one-to-one but not onto.
   (v) If $\dim(\mathrm{im}(T)) = \dim(W)$, then T is onto.
   (w) If V is isomorphic to W, then $\dim(V) = \dim(W)$.
   (x) If $S, T : V \to V$ then $[ST]_\alpha^\gamma = [S]_\alpha^\beta [T]_\beta^\gamma$ for any ordered bases α, β, γ of V.
   (y) If $T : V \to V$, then $[T^2\mathbf{v}]_\beta = \left([T]_\alpha^\beta\right)^2 [\mathbf{v}]_\alpha$ for any ordered bases α, β of V and $\mathbf{v} \in V$.
   (z) If $T : V \to V$ has an inverse T^{-1}, then $[T^{-1}]_\alpha^\beta = ([T]_\alpha^\beta)^{-1}$ for any ordered bases α, β of V.

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2. An $n \times n$ matrix A is <u>orthogonal</u> when $A^T A = I_n$. If A is orthogonal, show that A^{-1} is also orthogonal.

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3. Suppose A and B are $n \times n$ matrices such that B and ABA are invertible. Show that B is invertible.

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4. Prove that $1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^n} = 2 - \dfrac{1}{2^n}$ for all positive integers n.

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5. Prove that the nth power of $\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$ is $\begin{bmatrix} 2n+1 & 2n \\ -2n & 1-2n \end{bmatrix}$ for all positive integers n.

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6. Suppose A is an invertible upper-triangular matrix. Show that A^{-1} is also upper-triangular. [Hint: Consider A as a product of elementary row matrices.]

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7. Determine whether the given S is a subspace of the given vector space V and if so find its dimension:
   (a) $V = \mathbb{R}^4$, $S =$ the vectors $\langle a,b,c,d\rangle$ in $\mathbb{R}^4$ with $a + 2b + 3c + 4d = 5$.
   (b) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with nonnegative real entries.
   (c) $V =$ twice-differentiable functions on $[0,1]$, $S =$ the functions with $f''(x) + f(x) = 1$.
   (d) $V = \mathbb{C}^5$, $S =$ the vectors $\langle a,b,c,d,e\rangle$ with $e = a + b$ and $b = c = d$.
   (e) $V = P_3(\mathbb{C})$, $S =$ the polynomials with $p'(1) = 0$.

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8. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space V. Show $\{\mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2 - \mathbf{v}_1\}$ is also a basis for V.

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9. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ are two linearly-independent subsets of V.
   (a) If $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \cap \text{span}(\mathbf{w}_1, \mathbf{w}_2) = \{\mathbf{0}\}$, prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent.
   (b) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent, prove that $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \cap \text{span}(\mathbf{w}_1, \mathbf{w}_2) = \{\mathbf{0}\}$.

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10. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans V and that for each i, $\mathbf{v}_i$ is not contained in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n)$. Prove that S is a basis for V.

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11. Prove that $\dim(\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)) = \dim(\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{w}))$ if and only if $\mathbf{w}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$.

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12. Suppose that $\dim(V) = 3$, and that W_1 and W_2 are subspaces of V with $\dim(W_1) = \dim(W_2) = 2$. If $W_1 \neq W_2$, prove that $W_1 + W_2 = V$. (Recall $W_1 + W_2$ is the set of all vectors $\mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W_1 and $\mathbf{w}_2$ is in W_2.)

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13. For each linear $T : V \to W$, find bases for the kernel and image of T and find $[T]_\beta^\gamma$ for the given β, γ:
   (a) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $T(a, b, c) = (a - b, b - c)$, with $\beta = \{\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\}$, $\gamma = \{\langle 1, 0\rangle, \langle 0, 1\rangle\}$.
   (b) $V = P_2(\mathbb{C})$, $W = P_3(\mathbb{C})$, $T(p) = p(x) - xp'(x)$, with $\beta = \{1, x, x^2\}$, $\gamma = \{1, x, x^2, x^3\}$.
   (c) $V = W = M_{2\times 2}(\mathbb{R})$, $T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A - A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, with $\beta = \gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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14. Determine whether there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with the given kernel and image: if there is, find an example, and if not explain why not:
   (a) $\ker(T) = \{\langle a, a, a\rangle, a \in \mathbb{R}\}$ and $\text{im}(T) = \mathbb{R}^2$.
   (b) $\ker(T) = \{\langle a, a, a\rangle, a \in \mathbb{R}\}$ and $\text{im}(T) = \{\langle c, 0\rangle, c \in \mathbb{R}\}$.
   (c) $\ker(T) = \{\langle a, b, a + b\rangle, a, b \in \mathbb{R}\}$ and $\text{im}(T) = \{\langle c, 0\rangle, c \in \mathbb{R}\}$.

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15. Suppose $T : V \to W$ is linear where V and W are finite-dimensional and $\dim V < \dim W$. Show that T is not onto.

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16. Suppose $T : V \to W$ is a linear transformation where $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V. If $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a basis of W, prove that T is an isomorphism.

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17. Suppose V and W are finite-dimensional vector spaces and $T : V \to W$ is linear, β is an ordered basis of V, and γ is an ordered basis of W.
   (a) If $[T]_\beta^\gamma$ is the identity matrix, show that T is an isomorphism.
   (b) If T is an isomorphism, show β and γ can be chosen so that $[T]_\beta^\gamma$ is the identity matrix.

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18. Suppose $T : V \to V$ is a linear transformation.
   (a) If $\ker(T) = \ker(T^2)$, show that $\text{im}(T) \cap \ker(T) = \{\mathbf{0}\}$.
   (b) If $\text{im}(T) = \text{im}(T^2)$ and V is finite-dimensional, show that $\text{im}(T) \cap \ker(T) = \{\mathbf{0}\}$.
   (c) Show that the derivative map $D : \mathbb{R}[x] \to \mathbb{R}[x]$ has $\text{im}(D) = \text{im}(D^2)$, but also has $\text{im}(D) \cap \ker(D)$ containing a nonzero polynomial.
