1. (a) True: direct check
   (b) False: can't have > 2 independent vectors
   (c) False: need at least 4 to span
   (d) True: polynomials have different degrees
   (e) True: extend a basis of W to a basis of V
   (f) True: by definition of finite-dimensional
   (g) False: "all vectors in V" is not independent
   (h) False: take $\{1, x, 0\}$ in $P_2(\mathbb{R})$.
   (i) True: for instance, a basis works
   (j) False: take 2 vectors of a basis
   (k) False: the zero space has dimension 0
   (l) True: for $f = e^{-x} - 2e^x$ notice $f'' - f = 0$
   (m) False: cannot have $a - b = 2$, $a - b = 4$.
   (n) True: $T(2x) = x^2$
   (o) True: this is the definition of one-to-one
   (p) False: condition is backwards; onto means "for any $\mathbf{w} \in W$, exists $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$"
   (q) True: condition says T is a bijection
   (r) False: nullity-rank forces domain to have dim 4
   (s) True: $T(a, b, c, d, e) = (a, 0, 0, 0)$
   (t) True: $T(a, b, c, d, e) = (a, b, c, d)$
   (u) False: $\dim(\operatorname{im} T) \le 4$ so $\dim(\ker T) \ge 1$
   (v) False: this is only true if W is finite-dimensional
   (w) True: isomorphisms preserve dimension
   (x) False: bases are in the wrong order for composition
   (y) False: can't compose $\alpha \to \beta$ twice in a row
   (z) False: inverse reverses the input and output bases.

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2. If A is orthogonal then $A^{-1} = A^T$ so transposing gives $(A^{-1})^T = A = (A^{-1})^{-1}$ so A^{-1} is orthogonal.

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3. Taking determinants we see $\det(B)$ is nonzero and $\det(ABA) = \det(A)^2 \det(B)$ is nonzero, so $\det(A)^2$ hence $\det(A)$ is nonzero, so A is invertible.

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4. Induct on n. Base case $n = 1$ clear. Inductive step: If $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$, then $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}$.

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5. Induct on n. Base case $n = 1$ clear. Inductive step:
if $A^n = \begin{bmatrix} 2n+1 & 2n \\ -2n & 1-2n \end{bmatrix}$ then $A^{n+1} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2n+1 & 2n \\ -2n & 1-2n \end{bmatrix} = \begin{bmatrix} 2n+3 & 2n+2 \\ -2n-2 & -2n+1 \end{bmatrix}$ as required.

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6. Consider A as a product of elementary row matrices obtained by row-reducing A to I_n. Since A is upper-triangular and must have nonzero diagonal entries, each row-reduction involves subtracting a multiple of a lower row from a higher one, so the elementary matrices are all upper-triangular. Their product A^{-1} is then upper-triangular.

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7. (a) Not a subspace (does not contain zero vector)
   (b) Not a subspace (not closed under negative scalings)
   (c) Not a subspace (does not contain zero vector)
   (d) Subspace (satisfies [S1]-[S3]). Basis $\{\langle 1, 0, 0, 0, 1\rangle, \langle 0, 1, 1, 1, 1\rangle\}$.
   (e) Subspace (kernel of $T : V \to \mathbb{C}$ with $T(p) = p'(1)$). Basis $\{1, x^2 - 2x, x^3 - 3x\}$.

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8. Span: Observe $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = (c - b)\mathbf{v}_3 + (a - b)(\mathbf{v}_3 - \mathbf{v}_2) + (-a)(\mathbf{v}_3 - \mathbf{v}_2 - \mathbf{v}_1)$. Independence: If $a\mathbf{v}_3 + b(\mathbf{v}_3 - \mathbf{v}_2) + c(\mathbf{v}_3 - \mathbf{v}_2 - \mathbf{v}_1) = \mathbf{0}$ then $(a + b + c)\mathbf{v}_3 - (b + c)\mathbf{v}_2 - c\mathbf{v}_1 = 0$ so $a + b + c = b + c = c = 0$ so $a = b = c = 0$.

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9. (a) If $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + b_1\mathbf{w}_1 + b_2\mathbf{w}_2 = \mathbf{0}$ then let $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = -b_1\mathbf{w}_1 - b_2\mathbf{w}_2$. Then $\mathbf{x}$ is in $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \cap \operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$ hence is zero. But then since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ are independent the two expressions for $\mathbf{x}$ imply $a_1 = a_2 = a_3 = b_1 = b_2 = 0$.
   (b) If $\mathbf{x} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \cap \operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$ suppose $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = b_1\mathbf{w}_1 + b_2\mathbf{w}_2$. Subtracting, $\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$: as $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent this requires $a_1 = a_2 = a_3 = b_1 = b_2 = 0$ so then $\mathbf{x} = \mathbf{0}$.

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10. Suppose S had a linear dependence $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ with $a_i \neq 0$. Then solving for $\mathbf{v}_i$ yields $\mathbf{v}_i = (-1/a_i)(a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_n\mathbf{v}_n)$, contradicting the hypothesis. So all coefficients must be zero, so S is linearly independent.

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11. If $\mathbf{w}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{w})$ as proven in class, so the dimensions are equal. Otherwise, if $\mathbf{w}$ is not in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{w})$ is strictly larger than $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ so its dimension is also strictly larger.

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12. If $W_1 \neq W_2$ then W_2 must contain some vector $\mathbf{v}$ not in W_1 (otherwise $W_2 \subseteq W_1$ but since they both have dimension 2, they would be equal). Then $W_1 + W_2$ contains both W_1 and $\mathbf{v}$ so its dimension is strictly larger than that of W_1: then since $\dim(V) = 3$ this means $\dim(W_1 + W_2) = 3$ so $W_1 + W_2 = V$.

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13. (a) Kernel basis $\{\langle 1,1,1\rangle\}$, image basis $\{\langle 1,0\rangle, \langle 0,1\rangle\}$, $[T]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

(b) Kernel basis $\{x\}$, image basis $\{1, x^2\}$, $[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

(c) Kernel basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, image basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, $[T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

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14. (a) Many examples, such as $T(x,y,z) = (x-y, y-z)$.

(b) Impossible by nullity-rank since $\dim(\ker T) = 1 = \dim(\text{im}\, T)$.

(c) Many examples, such as $T(x,y,z) = (x+y-z, 0)$.

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15. By the nullity-rank theorem, we have $\dim(\text{im}\, T) \leq \dim(\ker T) + \dim(\text{im}\, T) = \dim V < \dim W$ so $\text{im}(T)$ cannot equal W hence T is not onto.

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16. Solution 1: Let $\mathbf{w}_i = T(\mathbf{v}_i)$. Since $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a basis of W, by properties of linear transformations there exists a linear transformation $S : W \to V$ with $S(\mathbf{w}_i) = \mathbf{v}_i$ for each i. Then S and T are inverses on the bases $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$: thus T has an inverse so it is an isomorphism.
Solution 2: Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\gamma = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$. Then $[T]_\beta^\gamma$ is the identity matrix, which is invertible, so T is an isomorphism.

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17. (a) By properties of associated matrices, since $[T]_\beta^\gamma$ is invertible, the associated transformation is an isomorphism with inverse transformation $S : W \to V$ having matrix $[S]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

(b) Take any basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and let $\gamma = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$: since T is an isomorphism, it maps a basis of V to a basis of W, so γ is a basis of W. Then $[T]_\beta^\gamma = I_n$ directly by definition of the associated matrix.

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18. (a) If $\ker(T) = \ker(T^2)$ let $\mathbf{w} \in \text{im}(T) \cap \ker(T)$. Then $\mathbf{w} = T(\mathbf{v})$ so then $\mathbf{0} = T(\mathbf{w}) = T^2(\mathbf{v})$: thus $\mathbf{v} \in \ker(T^2)$ hence $\mathbf{v} \in \ker(T)$ and thus $\mathbf{w} = T(\mathbf{v}) = \mathbf{0}$.

(b) If $\text{im}(T) = \text{im}(T^2)$ then $\dim(V) - \dim(\text{im}(T)) = \dim(V) - \text{im}(T^2)$ hence by nullity-rank this yields $\dim(\ker(T)) = \dim(\ker(T^2))$. But since $\ker(T) \subseteq \ker(T^2)$, since $T(\mathbf{v}) = \mathbf{0}$ implies $T^2(\mathbf{v}) = T(\mathbf{0}) = \mathbf{0}$, and these spaces are finite-dimensional, we see $\ker(T) = \ker(T^2)$. Then by (a), $\text{im}(T) \cap \ker(T) = \{\mathbf{0}\}$.

(c) The derivative map is onto so $\text{im}(D) = \text{im}(D^2) = \mathbb{R}[x]$, but nonzero constants are also in $\text{im}(D) \cap \ker(D)$.
