Math 4571: Advanced Linear Algebra

Midterm 1 (Instructor: Dummit)

February 18th, 2026

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 7 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	16	
2	12	
3	8	
4	8	
5	8	
6	8	
7	8	
8	12	
TOTAL	80	

1. (16 points) For each of the following, circle the correct response (there is no partial credit nor penalty for wrong answers). Assume $T : V \to W$ is a linear transformation, where V and W are **not necessarily finite-dimensional**.

True **False** If A is a 4×4 matrix whose first three rows are equal, then A is invertible.

True **False** The polynomials $1 + x$, $2 + x^2$, $3 + x^3$ are linearly independent.

True **False** The matrices $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ span $M_{2\times 2}(\mathbb{R})$.

True **False** If W is a subspace of V, there is a basis for V containing a basis for W.

True **False** If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a basis of $\text{im}(T)$.

True **False** The nullspace of the real matrix $\begin{bmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$ has dimension 3.

True **False** The linear transformation $T : M_{2\times 3}(\mathbb{R}) \to M_{3\times 2}(\mathbb{R})$ given by $T(A) = 2A^T$ is one-to-one and onto.

True **False** There is a vector space V and a linear transformation $T : V \to V$ that is onto but not an isomorphism.

True **False** Any two finite-dimensional vector spaces having the same dimension and scalar field are isomorphic.

Now assume that the vector spaces V and W are finite-dimensional, that α, β, and γ are ordered bases of V, V, and W respectively, and that S and T are linear transformations.

True **False** If $S(\mathbf{v}) = T(\mathbf{v})$ for all vectors $\mathbf{v}$ in an ordered basis β of V, then $S = T$.

True **False** If $I : V \to V$ is the identity map, then $[I]_\alpha^\beta$ is always the identity matrix.

True **False** If $S : V \to W$ and $T : W \to V$, then $[ST]_\gamma^\gamma = [S]_\beta^\gamma [T]_\gamma^\beta$.

2. (12 points) Let $S : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation $S(A) = A + A^T$.

(a) Find $[S]_\beta^\beta$ for the standard basis $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(b) Prove that $S^{n+1} = 2^n S$ for every positive integer n.

(c) Find a basis for $\ker(S)$ and for $\operatorname{im}(S)$.

3. (8 points) A real $n \times n$ matrix A is orthogonal when $A^T = A^{-1}$.

(a) If A is an orthogonal $n \times n$ matrix, show that $\det(A) = 1$ or -1.

(b) If A and B are orthogonal $n \times n$ matrices, show that AB is orthogonal.

4. (8 points) Suppose $T : V \to W$ is linear. If W_1 is a subspace of W, we define the inverse image $T^{-1}(W_1)$ to be all vectors in V whose image under T lies in W_1: explicitly, $T^{-1}(W_1) = \{\mathbf{v} \in V \, : \, T(\mathbf{v}) \in W_1\}$. Show that $T^{-1}(W_1)$ is a subspace of V.
[Warning: Note that T^{-1} is not necessarily a function.]

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5. (8 points) Suppose $T : V \to W$ is a linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of V. If $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent, show that $\ker(T) = \{\mathbf{0}\}$.

6. (8 points) Suppose $T : V \to W$ is a linear transformation where $\dim(V) = 2026$ and $\dim(W) = 1900$. Prove that there exists a linearly independent subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{100}\}$ of V such that $T(\mathbf{v}_i) = \mathbf{0}$ for each $1 \le i \le 100$.

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7. (8 points) Let V be a (not necessarily finite-dimensional) vector space and $T : V \to V$ be linear and such that T^3 is the identity transformation. Prove that T is an isomorphism.

8. (12 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that T^2 is the zero transformation, but T is not.

(a) Show that the kernel of T contains the image of T.

(b) Show that $\dim(\mathrm{im}(T)) = 1$.

(c) Let $\mathbf{v}$ be a nonzero vector in $\mathrm{im}(T)$, where $T(\mathbf{w}) = \mathbf{v}$. Prove that $\beta = \{\mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbb{R}^2$. [Hint: Apply T to a linear dependence.]

(d) With $\beta = \{\mathbf{v}, \mathbf{w}\}$ as in part (c), show that the matrix $[T]_\beta^\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.