Math 4571 (Advanced Linear Algebra)

Lecture #8 of 38 $\sim$ January 26, 2026

Bases and Dimension

- Existence of Bases
- Properties of Bases
- Dimension

This material represents §1.5.1-§1.5.3 from the course notes.

Recall

Recall the notions of span and linear independence from last week:

Definition

The span of a set S of vectors is the set of all linear combinations of (finitely many) vectors in S: namely, all $\mathbf{w}$ of the form $\mathbf{w} = a_1 \cdot \mathbf{v}_1 + \cdots + a_n \cdot \mathbf{v}_n$, for scalars $a_1, \ldots, a_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in S$.

Definition

A set S of vectors is linearly independent when $a_1 \cdot \mathbf{v}_1 + \cdots + a_n \cdot \mathbf{v}_n = \mathbf{0}$ implies $a_1 = \cdots = a_n = 0$ for any (distinct) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in S$.

Bases

Definition

A linearly independent set of vectors that spans V is called a basis for V.

Having a basis allows us to describe all the elements of a vector space in a particularly convenient way:

Proposition (Characterization of Bases)

The set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ forms a basis of the vector space V if and only if every vector $\mathbf{w}$ in V can be written in the form $\mathbf{w} = a_1 \cdot \mathbf{v}_1 + a_2 \cdot \mathbf{v}_2 + \cdots + a_n \cdot \mathbf{v}_n$ for unique scalars $a_1, a_2, \ldots, a_n$.

One should think of the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \ldots$ as "coordinate directions" and the coefficients $a_1, a_2, \ldots, a_n, \ldots$ as "coordinates". (Even when the basis is infinite, only finitely many of the basis vectors are needed to reach any vector.)

Bases from Spanning Sets, I

A basis for a vector space can be obtained from a spanning set:

Theorem (Spanning Sets and Bases)

If V is a vector space, then any set spanning V contains a basis of V.

In the event that the spanning set is infinite, the argument is rather delicate and technical (and requires the axiom of choice), and does not give such good intuition. Thus, we will instead give the proof of this theorem in the special case where the spanning set is finite.

Bases from Spanning Sets, II

Proof (finite spanning set case):

- Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V.
- Start with an empty collection S_0 of elements and now, for each $1 \leq k \leq n$, perform the following procedure:
  1. Check whether the vector $\mathbf{v}_k$ is contained in the span of S_{k-1}.
  2. If $\mathbf{v}_k$ is not in the span of S_{k-1}, let $S_k = S_{k-1} \cup \{\mathbf{v}_k\}$. Otherwise, let $S_k = S_{k-1}$.
- We claim that the set S_n, obtained at the end of this procedure, is a basis for V.
- We will show that each set S_k is linearly independent, and the span of S_n is the same as the span of S,

Bases from Spanning Sets, III

1. Start with S_0 empty. Then for each k:
2. If $\mathbf{v}_k$ is not in the span of S_{k-1}, let $S_k = S_{k-1} \cup \{\mathbf{v}_k\}$. Otherwise, let $S_k = S_{k-1}$.

Claim: The set S_k is linearly independent, for each $0 \leq k \leq n$.

- Base Case ($k = 0$): clearly, S_0 is linearly independent.
- Inductive Step: Suppose $k \geq 1$ and S_{k-1} is linearly independent.
- If $\mathbf{v}_k$ is in the span of S_{k-1}, then $S_k = S_{k-1}$ is linearly independent.
- If $\mathbf{v}_k$ is not in the span of S_{k-1}, then $S_k = S_{k-1} \cup \{\mathbf{v}_k\}$ is linearly independent by our proposition about span and linear independence.

Bases from Spanning Sets, IV

1. Start with S_0 empty. Then for each k:
2. If $\mathbf{v}_k$ is not in the span of S_{k-1}, let $S_k = S_{k-1} \cup \{\mathbf{v}_k\}$. Otherwise, let $S_k = S_{k-1}$.

Claim: For $T_k = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, we have $\mathrm{span}(S_k) = \mathrm{span}(T_k)$ for each $0 \leq k \leq n$.

- Base Case ($k = 0$): Trivial, as both S_0 and T_0 are empty.
- Inductive Step: Suppose $k \geq 1$ and $\mathrm{span}(S_{k-1}) = \mathrm{span}(T_{k-1})$
- If $\mathbf{v}_k$ is in the span of S_{k-1}, then $S_{k-1} = S_k$.
- By hypothesis, $\mathrm{span}(S_{k-1}) = \mathrm{span}(T_{k-1})$ so $\mathbf{v}_k$ is in the span of T_{k-1}. By properties of span, $\mathrm{span}(T_k) = \mathrm{span}(T_{k-1})$, so $\mathrm{span}(T_k) = \mathrm{span}(S_k)$ as claimed.

Bases from Spanning Sets, IV

1. Start with S_0 empty. Then for each k:
2. If $\mathbf{v}_k$ is not in the span of S_{k-1}, let $S_k = S_{k-1} \cup \{\mathbf{v}_k\}$. Otherwise, let $S_k = S_{k-1}$.

<u>Claim</u>: For $T_k = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, we have $\mathrm{span}(S_k) = \mathrm{span}(T_k)$ for each $0 \le k \le n$.

- <u>Base Case</u> ($k = 0$): Trivial, as both S_0 and T_0 are empty.
- <u>Inductive Step</u>: Suppose $k \ge 1$ and $\mathrm{span}(S_{k-1}) = \mathrm{span}(T_{k-1})$
- If $\mathbf{v}_k$ is in the span of S_{k-1}, then $S_{k-1} = S_k$.
- By hypothesis, $\mathrm{span}(S_{k-1}) = \mathrm{span}(T_{k-1})$ so $\mathbf{v}_k$ is in the span of T_{k-1}. By properties of span, $\mathrm{span}(T_k) = \mathrm{span}(T_{k-1})$, so $\mathrm{span}(T_k) = \mathrm{span}(S_k)$ as claimed.
- If $\mathbf{v}_k$ is not in the span of S_{k-1}, then in a similar way, since $\mathrm{span}(S_{k-1}) = \mathrm{span}(T_{k-1})$, we have $\mathrm{span}(S_{k-1} \cup \{\mathbf{v}_k\}) = \mathrm{span}(S_{k-1} \cup \{\mathbf{v}_k\})$, which is the same as saying $\mathrm{span}(S_k) = \mathrm{span}(T_k)$.

Existence of Bases

Corollary

Every vector space V has a basis.

Proof: The set of all vectors in V spans V. By the previous result, this spanning set contains a basis.

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Proof: The set of all vectors in V spans V. By the previous result, this spanning set contains a basis.

A basis for a vector space can also be obtained from a linearly independent set:

Theorem (Building-Up Theorem)

Any linearly independent set of vectors can be extended to a basis.

As with the analogous theorem about spanning sets, proving this theorem in general requires the axiom of choice. Thus, we will instead give the proof of this theorem in the special case where the vector space has a finite basis.

The Replacement Theorem, I

The most convenient approach is using a preliminary result known as the replacement theorem.

Theorem (Replacement Theorem)

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V and $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ is a linearly independent subset of V. Then there is a reordering of the basis S, say $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ such that for each $0 \le k \le m$, the set $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k, \mathbf{a}_{k+1}, \mathbf{a}_{k+2}, \dots, \mathbf{a}_n\}$ is a basis for V.

Equivalently, the elements $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ can be used to successively replace the elements of the basis, with each replacement remaining a basis of V.

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Equivalently, the elements $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ can be used to successively replace the elements of the basis, with each replacement remaining a basis of V.

The proof of this theorem is not conceptually hard, but the actual details are annoyingly cumbersome to write down.

The Replacement Theorem, II

Theorem (Replacement Theorem)

Suppose S is a finite basis of V and $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ is linearly independent. Then there is an ordering $\{\mathbf{a}_1 \dots, \mathbf{a}_n\}$ of S such that $\{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\}$ is a basis of V for each $0 \le k \le n$.

Proof:

- We induct on k. The base case $k = 0$ is trivial, since it simply claims that $S = \{\mathbf{a}_1 \dots, \mathbf{a}_n\}$ is a basis, which is given.

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Suppose S is a finite basis of V and $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ is linearly independent. Then there is an ordering $\{\mathbf{a}_1 \dots, \mathbf{a}_n\}$ of S such that $\{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\}$ is a basis of V for each $0 \le k \le n$.

Proof:

- We induct on k. The base case $k = 0$ is trivial, since it simply claims that $S = \{\mathbf{a}_1 \dots, \mathbf{a}_n\}$ is a basis, which is given.
- For the inductive step, suppose that $B_k = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k, \mathbf{a}_{k+1}, \mathbf{a}_{k+2}, \dots, \mathbf{a}_n\}$ is a basis for V.
- We must show that we can remove one of the vectors $\mathbf{a}_i$ and reorder the others to produce a basis $B_{k+1} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \dots, \mathbf{a}'_n\}$ for V.

The Replacement Theorem, III

Given: $B_k = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n\}$ and $S = \{\mathbf{a}_1 \ldots, \mathbf{a}_n\}$ are bases and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}\}$ is linearly independent.

Goal: Construct a basis $B_{k+1} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \ldots, \mathbf{a}'_n\}$.

- By hypothesis, since B_k spans V, we can write $\mathbf{w}_{k+1} = c_1 \cdot \mathbf{w}_1 + \cdots + c_k\mathbf{w}_k + d_{k+1}\mathbf{a}_{k+1} + \cdots + d_n\mathbf{a}_n$ for some scalars c_i and d_i.

The Replacement Theorem, III

<u>Given</u>: $B_k = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n\}$ and $S = \{\mathbf{a}_1 \ldots, \mathbf{a}_n\}$ are bases and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}\}$ is linearly independent.

<u>Goal</u>: Construct a basis $B_{k+1} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \ldots, \mathbf{a}'_n\}$.

- By hypothesis, since B_k spans V, we can write
  $\mathbf{w}_{k+1} = c_1 \cdot \mathbf{w}_1 + \cdots + c_k \mathbf{w}_k + d_{k+1}\mathbf{a}_{k+1} + \cdots + d_n \mathbf{a}_n$
  for some scalars c_i and d_i.
- If all of the d_i were zero, then $\mathbf{w}_{k+1}$ would be a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_k$, contradicting the assumption that $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k+1}\}$ is linearly independent.
- Thus, at least one d_i is not zero. Rearrange the vectors $\mathbf{a}_i$ so that $d_{k+1} \neq 0$: then
  $\mathbf{w}_{k+1} = c_1 \mathbf{w}_1 + \cdots + c_k \mathbf{w}_k + d'_{k+1}\mathbf{a}'_{k+1} + \cdots + d'_n \mathbf{a}'_n$.
- We claim $B_{k+1} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \ldots, \mathbf{a}'_n\}$ is a basis for V. (For convenience, drop the primes on the vectors.)

The Replacement Theorem, IV

<u>Given</u>: $B_k = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n\}$ and $S = \{\mathbf{a}_1 \ldots, \mathbf{a}_n\}$ are bases and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}\}$ is linearly independent, and $\mathbf{w}_{k+1} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k + d_{k+1}\mathbf{a}_{k+1} + \cdots + d_n\mathbf{a}_n$ with $d_{k+1} \neq 0$.

<u>Goal</u>: Show $B_{k+1} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}_{k+2}, \ldots, \mathbf{a}_n\}$ is a basis.

- First, B_{k+1} spans V.

The Replacement Theorem, IV

<u>Given</u>: $B_k = \{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\}$ and $S = \{\mathbf{a}_1 \dots, \mathbf{a}_n\}$ are bases and $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k, \mathbf{w}_{k+1}\}$ is linearly independent, and $\mathbf{w}_{k+1} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k + d_{k+1}\mathbf{a}_{k+1} + \cdots + d_n\mathbf{a}_n$ with $d_{k+1} \neq 0$.

<u>Goal</u>: Show $B_{k+1} = \{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}_{k+2}, \dots, \mathbf{a}_n\}$ is a basis.

- First, B_{k+1} spans V.
- Since $d_{k+1} \neq 0$, we can solve for $\mathbf{a}'_{k+1}$ as a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \dots, \mathbf{a}'_n$. (Write it down if you like!)
- If $\mathbf{x}$ is any vector in V, since B_k spans V we can write $\mathbf{x} = e_1 \cdot \mathbf{w}_1 + \cdots + e_k \cdot \mathbf{w}_k + e_{k+1} \cdot \mathbf{a}'_{k+1} + \cdots + e_n \cdot \mathbf{a}'_n$.
- Plugging in the expression for $\mathbf{a}'_{k+1}$ in terms of $\mathbf{w}_1, \dots, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \dots, \mathbf{a}'_n$ then shows that $\mathbf{x}$ is a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \dots, \mathbf{a}'_n$.

The Replacement Theorem, V

Given: $B_k = \{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\}$ and $S = \{\mathbf{a}_1 \dots, \mathbf{a}_n\}$ are bases and $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k, \mathbf{w}_{k+1}\}$ is linearly independent, and $\mathbf{w}_{k+1} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k + d_{k+1}\mathbf{a}_{k+1} + \cdots + d_n\mathbf{a}_n$ with $d_{k+1} \neq 0$.

Goal: Show $B_{k+1} = \{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}_{k+2}, \dots, \mathbf{a}_n\}$ is a basis.

- Second: B_{k+1} is linearly independent.

The Replacement Theorem, V

<u>Given</u>: $B_k = \{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\}$ and $S = \{\mathbf{a}_1 \dots, \mathbf{a}_n\}$ are bases and $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k, \mathbf{w}_{k+1}\}$ is linearly independent, and $\mathbf{w}_{k+1} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k + d_{k+1}\mathbf{a}_{k+1} + \cdots + d_n\mathbf{a}_n$ with $d_{k+1} \neq 0$.

<u>Goal</u>: Show $B_{k+1} = \{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{w}_{k+1}, \mathbf{a}_{k+2}, \dots, \mathbf{a}_n\}$ is a basis.

- Second: B_{k+1} is linearly independent.
- Suppose we had a dependence $\mathbf{0} = f_1\mathbf{w}_1 + \cdots + f_k\mathbf{w}_k + f_{k+1}\mathbf{a}'_{k+1} + \cdots + f_n\mathbf{a}'_n$.
- Plug in the expression for $\mathbf{a}'_{k+1}$ in terms of $\mathbf{w}_1, \dots, \mathbf{w}_{k+1}, \mathbf{a}'_{k+2}, \dots, \mathbf{a}'_n$: all of the coefficients must be zero because B_k is linearly independent. But the coefficient of $\mathbf{w}_{k+1}$ is f_{k+1} times a nonzero scalar, so $f_{k+1} = 0$.
- This implies $\mathbf{0} = f_1\mathbf{w}_1 + \cdots + f_k\mathbf{w}_k + f_{k+2}\mathbf{a}'_{k+2} + \cdots + f_n\mathbf{a}'_n$, and this is a dependence involving the vectors in B_k.
- Since B_k is linearly independent, all coefficients are zero. Thus $f_1 = f_2 = \cdots = f_n = 0$ so B_{k+1} is linearly independent.

Applications of the Replacement Theorem, I

Although the proof of the Replacement Theorem is cumbersome, we obtain several useful corollaries.

Proposition (Corollaries of Replacement Theorem)

Let V be a vector space.

1. *If has a basis with n elements, then any set of $m > n$ vectors of V is linearly dependent.*
2. *Any two bases of a vector space have the same number of elements.*
3. *Any linearly independent set of vectors in V can be extended to a basis.*

Applications of the Replacement Theorem, II

1. If has a basis with n elements, then any set of $m > n$ vectors of V is linearly dependent.

Proof:

Applications of the Replacement Theorem, II

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Proof:

- Suppose $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ is a linearly independent subset of V and that V has a basis with n elements.
- Apply the Replacement Theorem with the given basis of V: at the nth step we have replaced all the elements of the original basis with those in our new set, so by the conclusion of the theorem we see that $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a basis for V.
- Then $\mathbf{w}_{n+1}$ is necessarily a linear combination of $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$, meaning that $\{\mathbf{w}_1, \dots, \mathbf{w}_n, \mathbf{w}_{n+1}\}$ is linearly dependent, and so $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ is too. Contradiction.

Applications of the Replacement Theorem, III

2. Any two bases of a vector space have the same number of elements.

Proof:

[1] There are different infinite cardinalities, and with more effort one can prove that even in the infinite case, the cardinalities of any two bases must actually be the same.

[2] We view the size of a basis as either a nonnegative integer or ∞. A basis of smallest size must exist by the well-ordering principle: any nonempty set of nonnegative integers has a smallest element.

Applications of the Replacement Theorem, III

2. Any two bases of a vector space have the same number of elements.

Proof:

- If every basis is infinite[1], we are already done. Else, V has some finite basis: choose B to be a basis of minimal size[2].
- Suppose B has n elements, and consider any other basis B' of V. By (1), if B' contains more than n vectors, it would be linearly dependent (impossible). Thus, B' also has n elements, so every basis of V has n elements.

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[2] We view the size of a basis as either a nonnegative integer or ∞. A basis of smallest size must exist by the well-ordering principle: any nonempty set of nonnegative integers has a smallest element.

Applications of the Replacement Theorem, IV

3. Any linearly independent set of vectors in V can be extended to a basis.

Proof (finite basis case):

Applications of the Replacement Theorem, IV

3. Any linearly independent set of vectors in V can be extended to a basis.

Proof (finite basis case):

- Let S be any set of linearly independent vectors and let B be any finite basis of V.
- Apply the Replacement Theorem to B and S: this produces a new basis of V containing S.

This particular result is often called the Building-Up Theorem.

Applications of the Replacement Theorem, V

We can also give a more constructive argument for extending a linearly independent set to a basis, similar to the algorithm used to reduce a spanning set to a basis.

Start with a linearly independent set S of vectors in V.

1. Check whether S spans V: if so, we are done.
2. If not, then there is an element $\mathbf{v}$ in V which is not in the span of S. Append $\mathbf{v}$ to S: then by our properties of linear independence, the new S will still be linearly independent.
3. Repeat the above two steps until S spans V.

Applications of the Replacement Theorem, V

We can also give a more constructive argument for extending a linearly independent set to a basis, similar to the algorithm used to reduce a spanning set to a basis.

Start with a linearly independent set S of vectors in V.

1. Check whether S spans V: if so, we are done.
2. If not, then there is an element $\mathbf{v}$ in V which is not in the span of S. Append $\mathbf{v}$ to S: then by our properties of linear independence, the new S will still be linearly independent.
3. Repeat the above two steps until S spans V.

If V possesses a basis of cardinality n, then this procedure will always terminate in at most n steps, since any set of more than n vectors would be linearly dependent. If V has an infinite basis, then one instead needs to make a Zorn's lemma argument to justify that this procedure constructs a basis if continued "forever".

So You Really Want To Hear About Zorn's Lemma, I

For completeness, we can give a brief sketch of the Zorn's lemma argument (for more details consult the challenge problem on homework 3).

- Zorn's lemma[3] states that if $\mathcal{F}$ is a nonempty partially-ordered set[4] in which every chain[5] has an upper bound[6], then $\mathcal{F}$ contains a maximal element[7].

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[3] Zorn's lemma is logically equivalent to the axiom of choice, which states that the Cartesian product of any collection of nonempty sets is nonempty.

[4] A partial ordering on a set is a relation $\leq$ such that $X \leq X$ for all X, if $X \leq Y$ and $Y \leq X$ then $X = Y$, and where $X \leq Y$ and $Y \leq Z$ imply $X \leq Z$.

[5] A chain is a totally ordered subset of $\mathcal{F}$, in which any two elements X and Y are comparable, meaning that either $X \leq Y$ or $Y \leq X$.

[6] An upper bound on a subset is an element $U \in \mathcal{F}$ such that $X \leq U$ for all X in the subset.

[7] A maximal element is an element $M \in \mathcal{F}$ such that if $M \leq Y$ for some $Y \in \mathcal{F}$, then in fact $Y = M$: no element is "above" M.

So You Really Want To Hear About Zorn's Lemma, II

Theorem

Every vector space V has a basis.

Proof (first half):

- Let $\mathcal{F}$ be the set of all linearly-independent subsets of V, partially ordered under inclusion. Since the empty set is linearly independent, $\mathcal{F}$ is nonempty.
- If $\mathcal{C}$ is any chain of $\mathcal{F}$, we claim $U = \bigcup_{A \in \mathcal{C}} A$ is an upper bound for $\mathcal{C}$ lying in $\mathcal{F}$. Clearly U is a subset of V so we just need to show it is linearly independent.
- Any dependence could involve only finitely many vectors, each of which is from some set in the chain. The maximum of these sets in the chain contains all the vectors, so since this set is linearly independent, we would get a contradiction.

So You Really Want To Hear About Zorn's Lemma, III

Theorem

Every vector space V has a basis.

Proof (second half):

- So now, by Zorn's lemma, since every chain of $\mathcal{F}$ has an upper bound, there exists some maximal element S of $\mathcal{F}$.
- This S is a linearly independent subset of V that is maximal under inclusion with respect to being linearly independent.
- If S didn't span V, there would exist some $\mathbf{w} \in V$ with $\mathbf{w} \notin \mathrm{span}(S)$. But then $S \cup \{\mathbf{w}\}$ would also be linearly independent, contradicting the maximality of S.
- Thus, S spans V and is linearly independent, so it is a basis.

Dimension, I

As we have just shown, any two bases of a vector space have the same number of elements. This quantity (the cardinality of a basis) is of fundamental importance:

Definition

If V is a vector space, the number of elements in any basis of V is called the dimension of V and is denoted $\dim(V)$.

If the dimension of V is a finite number, we say that V is finite-dimensional; otherwise, we say V is infinite-dimensional.

Our results above assure us that the dimension of a vector space is always well-defined: every vector space has a basis, and any other basis will have the same number of elements.

Dimension, II

Examples:

1. The dimension of F^n is n, since the n standard unit vectors $\mathbf{e}_i$ (with a 1 in the ith coordinate and 0s elsewhere) give a basis.

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2. The dimension of $M_{m\times n}(F)$ is mn, because the mn matrices $E_{i,j}$ (with a 1 in the (i,j)-entry and 0s elsewhere) give a basis.

Dimension, II

<u>Examples</u>:

1. The dimension of F^n is n, since the n standard unit vectors $\mathbf{e}_i$ (with a 1 in the ith coordinate and 0s elsewhere) give a basis.
2. The dimension of $M_{m\times n}(F)$ is mn, because the mn matrices $E_{i,j}$ (with a 1 in the (i,j)-entry and 0s elsewhere) give a basis.
3. The dimension of $F[x]$ is ∞, because the (infinite list of) polynomials $1, x, x^2, x^3, \dots, x^n, \dots$ is a basis.

Dimension, II

<u>Examples</u>:

1. The dimension of F^n is n, since the n standard unit vectors $\mathbf{e}_i$ (with a 1 in the ith coordinate and 0s elsewhere) give a basis.
2. The dimension of $M_{m \times n}(F)$ is mn, because the mn matrices $E_{i,j}$ (with a 1 in the (i, j)-entry and 0s elsewhere) give a basis.
3. The dimension of $F[x]$ is ∞, because the (infinite list of) polynomials $1, x, x^2, x^3, \dots, x^n, \dots$ is a basis.
4. The dimension of $P_n(F)$ is $n + 1$, since the polynomials $1, x, x^2, \dots, x^n$ give a basis.

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Dimension, III

There is a bit of subtlety in our notion of dimension, in that the choice of scalar field matters. Contrast these two examples:

- With scalar field $\mathbb{R}$, the vector space $\mathbb{C}$ has dimension 2, since $\{1, i\}$ is a basis.

Dimension, III

There is a bit of subtlety in our notion of dimension, in that the choice of scalar field matters. Contrast these two examples:

- With scalar field $\mathbb{R}$, the vector space $\mathbb{C}$ has dimension 2, since $\{1, i\}$ is a basis.
- With scalar field $\mathbb{C}$, the vector space $\mathbb{C}$ has dimension 1, since $\{1\}$ is a basis.

For this reason, one should indicate what the scalar field is when discussing dimension. We will write $\dim_F V$ for the dimension of V as a vector space over the field F, when F is not already clear from the context.

Dimension, IV

Example: Find the dimension of the subspace of $P_4(\mathbb{R})$ consisting of the polynomials with $p(1) = 0$. (The scalar field is $\mathbb{R}$.)

Dimension, IV

<u>Example</u>: Find the dimension of the subspace of $P_4(\mathbb{R})$ consisting of the polynomials with $p(1) = 0$. (The scalar field is $\mathbb{R}$.)

- The polynomials in $P_3(\mathbb{R})$ are of the form $p = ax^4 + bx^3 + cx^2 + dx + e$ for real numbers a, b, c, d, e.
- The condition $p(0) = 0$ yields $a + b + c + d + e = 0$ so that $e = -a - b - c - d$.
- So, the subspace consists of the polynomials of the form $ax^4 + bx^3 + cx^2 + dx - a - b - c - d =$ $a(x^4 - 1) + b(x^3 - 1) + c(x^2 - 1) + d(x - 1)$.
- Thus, the set $\{x^4 - 1, x^3 - 1, x^2 - 1, x - 1\}$ spans this subspace, but since it is also clearly linearly independent, it is a basis. Then the subspace has dimension $\boxed{4}$.

Dimension, V

Example: Find $\dim_{\mathbb{C}}(W)$ where W is the vector space of complex skew-symmetric 3×3 matrices A (i.e., with $A^T = -A$).

Dimension, V

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If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is skew-symmetric, then comparing A^T to $-A$ yields $a = e = i = 0$, $b = d$, $c = g$, and $h = f$. Then

$$A = b \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Since every matrix in the subspace can be uniquely decomposed as a linear combination of these three matrices, they yield a basis. The dimension is therefore $\boxed{3}$.

Infinite-Dimensional Things, I

Some properties of vector spaces depend on whether the space is finite-dimensional or infinite-dimensional.

- In general, finite-dimensional vector spaces are better behaved than infinite-dimensional vector spaces.
- We will therefore usually focus our attention on finite-dimensional spaces, since infinite-dimensional spaces can have occasional counterintuitive properties. (You'll get to see some of these on the homework as we progress further.)

Infinite-Dimensional Things, II

Example: The dimension of the vector space of all real-valued functions on the interval $[0, 1]$ is ∞, because it contains the linearly independent functions $1, x, x^2, x^3, \ldots$.

Infinite-Dimensional Things, II

Example: The dimension of the vector space of all real-valued functions on the interval $[0, 1]$ is ∞, because it contains the linearly independent functions $1, x, x^2, x^3, \ldots$.

- Notice that have not actually given a basis for this vector space, although (per our earlier results) this vector space does have a basis. There is a good reason we haven't: it is not possible to give a simple description of such a basis!
- For instance, the functions $f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$, for real numbers a, does not form a basis for the space of real-valued functions: although this infinite set is linearly independent, it does not span the space, since (for example) the constant function $f(x) = 1$ cannot be written as a finite linear combination of these functions.

Properties of Dimension, I

Here are some basic properties of dimension.

Proposition (Properties of Dimension)

Suppose V and W are vector spaces. Then the following hold:

1. *If W is a subspace of V, then $\dim(W) \leq \dim(V)$.*
2. *If $\dim(V) = n$, any linearly independent set of vectors has at most n elements.*
3. *If $\dim(V) = n$, any linearly independent set of n vectors is a basis for V.*
4. *If $\dim(V) = n$, any spanning set of V has at least n elements.*
5. *If $\dim(V) = n$, any spanning set of V having exactly n elements is a basis for V.*
6. *If $\dim(V) = n$ and S is a set of n vectors, then S is a basis if and only if it spans V if and only if it is linearly independent.*

Properties of Dimension, II

1. If W is a subspace of V, then $\dim(W) \leq \dim(V)$.

Proof:

Properties of Dimension, II

1. If W is a subspace of V, then $\dim(W) \leq \dim(V)$.

Proof: Choose any basis of W. It is a linearly independent set of vectors in V, so it is contained in some basis of V by the Building-Up Theorem.

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2. If $\dim(V) = n$, then any linearly independent set of vectors has at most n elements.
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2. If $\dim(V) = n$, then any linearly independent set of vectors has at most n elements.
3. If $\dim(V) = n$, then any linearly independent set of n vectors is a basis for V.

Proof: Both of these follow immediately from the Replacement Theorem: at the end of the replacement, we have a basis of n elements, and we have used all the vectors in our linearly independent set.

Properties of Dimension, III

4. If $\dim(V) = n$, then any spanning set of V has at least n elements.

Proof:

Properties of Dimension, III

4. If $\dim(V) = n$, then any spanning set of V has at least n elements.

Proof: As we showed, any spanning set contains a basis.

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5. If $\dim(V) = n$, then any spanning set of V having exactly n elements is a basis for V.

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5. If $\dim(V) = n$, then any spanning set of V having exactly n elements is a basis for V.

Proof: The spanning set contains a basis, but since the basis must also have n elements, the basis is the entire spanning set.

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6. If $\dim(V) = n$, a subset of V having exactly n vectors is a basis if and only if it spans V if and only if it is linearly independent.

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Properties of Dimension, III

4. If $\dim(V) = n$, then any spanning set of V has at least n elements.

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6. If $\dim(V) = n$, a subset of V having exactly n vectors is a basis if and only if it spans V if and only if it is linearly independent.

Proof: Basis implies both of the others. By (3), span implies basis, and by (5) linear independence implies basis. So all are equivalent.

More Examples, I

Example: Determine whether $\langle 1, 2, 2, 1\rangle$, $\langle 3, -1, 2, 0\rangle$, $\langle -3, 2, 1, 1\rangle$ span $\mathbb{R}^4$.

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Example: Determine whether $\langle 1, 2, 2, 1 \rangle$, $\langle 3, -1, 2, 0 \rangle$, $\langle -3, 2, 1, 1 \rangle$ span $\mathbb{R}^4$.

- $\boxed{\text{No}}$: since $\mathbb{R}^4$ is a 4-dimensional space, any spanning set must contain at least 4 vectors.

Example: Determine whether $\langle 1, 2, 1 \rangle$, $\langle 1, 0, 3 \rangle$, $\langle -3, 2, 1 \rangle$, $\langle 1, 1, 4 \rangle$ are linearly independent.

More Examples, I

Example: Determine whether $\langle 1,2,2,1\rangle$, $\langle 3,-1,2,0\rangle$, $\langle -3,2,1,1\rangle$ span $\mathbb{R}^4$.

- No: since $\mathbb{R}^4$ is a 4-dimensional space, any spanning set must contain at least 4 vectors.

Example: Determine whether $\langle 1,2,1\rangle$, $\langle 1,0,3\rangle$, $\langle -3,2,1\rangle$, $\langle 1,1,4\rangle$ are linearly independent.

- No: since $\mathbb{R}^3$ is a 3-dimensional space, any 4 vectors in $\mathbb{R}^3$ are automatically linearly dependent.

Any basis of F^n must have exactly n vectors. Not all sets of n vectors will give a basis, of course, so it would be nice if there were some simple way to *determine* whether n vectors give a basis...

More Examples, II

Proposition (Bases of F^n)

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are a basis of F^n if and only if $k = n$ and the matrix M with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is invertible.

From our analysis of determinants we know that M is invertible if and only if $\det(M) \neq 0$, so we simply need to compute a determinant to decide whether a given set of n vectors forms a basis.

More Examples, III

Claim: The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are a basis of F^n if and only if $k = n$ and the matrix M with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is invertible.

- Since $\dim(F^n) = n$ any basis must have n elements, so $k = n$.
- For any vector $\mathbf{w}$ in F^n, consider the problem of finding scalars $a_1, \cdots, a_n$ such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{w}$.

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- Since $\dim(F^n) = n$ any basis must have n elements, so $k = n$.
- For any vector $\mathbf{w}$ in F^n, consider the problem of finding scalars $a_1, \cdots, a_n$ such that $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{w}$.
- This vector equation is the same as the matrix equation $M\mathbf{a} = \mathbf{w}$, where M is the matrix whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, $\mathbf{a}$ is the column vector whose entries are the scalars $a_1, \ldots, a_n$, and $\mathbf{w}$ is viewed as a column vector.
- Then we have the following equivalences:
  The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ give a basis of F^n
  $\iff$ the scalars $a_1 \ldots, a_n$ are unique
  $\iff$ the system $M\mathbf{a} = \mathbf{w}$ always has a unique solution for $\mathbf{a}$
  $\iff$ the coefficient matrix M is invertible.

More Examples, IV

Example: Determine whether the vectors $\langle 1, 2, 1 \rangle$, $\langle 2, -1, 2 \rangle$, $\langle 3, 3, 1 \rangle$ form a basis of $\mathbb{R}^3$.

More Examples, IV

Example: Determine whether the vectors $\langle 1,2,1\rangle$, $\langle 2,-1,2\rangle$, $\langle 3,3,1\rangle$ form a basis of $\mathbb{R}^3$.

- By the proposition, we just need to find the determinant of $M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ whose columns are the given vectors.
- We compute $\det(M) = 1 \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 10$ which is nonzero.
- Thus, M is invertible, so these vectors $\boxed{\text{do form a basis of } \mathbb{R}^3}$.

Summary

We discussed the general existence of bases of vector spaces, and how to construct them using a spanning set and a linearly independent set.

We defined the dimension of a vector space and established some basic properties of dimension.

We computed some examples of dimensions and characterized bases of F^n.

Next lecture: Bases for the row space, column space, and nullspace of a matrix; linear transformations