1. Let V be a finite-dimensional vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

(a) If $\dim(V) = n$ and T has n distinct eigenvalues in F, then T is diagonalizable.

- $\boxed{\text{True}}$: if T has n distinct eigenvalues, then each eigenspace must have dimension 1. But then each eigenvalue's multiplicity is equal to the dimension of its eigenspace, so T is diagonalizable.

(b) If $\dim(V) = n$ and T is diagonalizable, then T has n distinct eigenvalues in F.

- $\boxed{\text{False}}$: there are diagonalizable linear transformations with repeated eigenvalues, such as the identity transformation (all its eigenvalues are 1, but it is clearly diagonalizable).

(c) If A is a diagonalizable $n \times n$ matrix, then so is $A + I_n$.

- $\boxed{\text{True}}$: if $Q^{-1}AQ = D$ is diagonal, then $Q^{-1}(A + I_n)Q = D + I_n$ is also diagonal.

(d) For any scalar λ, the λ-eigenspace of T is a subspace of the generalized λ-eigenspace of T.

- $\boxed{\text{True}}$: every λ-eigenvector is a generalized λ-eigenvector, so the λ-eigenspace is a subset (hence a subspace) of the generalized λ-eigenspace.

(e) For any λ, a chain of generalized λ-eigenvectors is linearly independent.

- $\boxed{\text{True}}$: we proved this in the course of showing that the generalized λ-eigenspace has a chain basis.

(f) There always exists a basis β of V consisting of generalized eigenvectors of T.

- $\boxed{\text{False}}$: we must also know that the eigenvalues of T all lie in the scalar field F. For example, the linear transformation $T(x, y) = (y, -x)$ has no such basis when $F = \mathbb{R}$, since its eigenvalues are $\pm i$.

(g) If all eigenvalues of T lie in F, then there exists a basis β of V of generalized eigenvectors for T.

- $\boxed{\text{True}}$: this was proven in class.

(h) There always exists some basis β of V such that the matrix $[T]_\beta^\beta$ is in Jordan canonical form.

- $\boxed{\text{False}}$: we must also know that the eigenvalues of T all lie in the scalar field F. For example, the linear transformation $T(x, y) = (y, -x)$ has no such basis when $F = \mathbb{R}$, since its eigenvalues are $\pm i$.

(i) Every matrix $A \in M_{n\times n}(\mathbb{C})$ has a Jordan canonical form.

- $\boxed{\text{True}}$: here, because the eigenvalues of A all lie in $\mathbb{C}$ (because $\mathbb{C}$ is algebraically closed), we know that A has a Jordan canonical form.

(j) If a matrix is diagonalizable, then its Jordan canonical form is diagonal.

- $\boxed{\text{True}}$: if a matrix is diagonalizable then its Jordan canonical form will be the diagonalization.

(k) If the Jordan canonical form of a matrix is diagonal, then the matrix is diagonalizable.

- $\boxed{\text{True}}$: since the matrix is similar to its Jordan form, that means the matrix is similar to a diagonal matrix, which is to say, it is diagonalizable.

(l) Two matrices are similar if and only if they have equivalent Jordan canonical forms.

- $\boxed{\text{True}}$: every matrix is similar to its Jordan canonical form, and similarity is transitive.

(m) If J is the Jordan canonical form of A, then $J + I_n$ is the Jordan canonical form of $A + I_n$.

- $\boxed{\text{True}}$: if $PAP^{-1} = J$ then $P(A + I_n)P^{-1} = J + I_n$, and $J + I_n$ is also in Jordan canonical form.

(n) If J is the Jordan canonical form of A, then J^2 is the Jordan canonical form of A^2.

- $\boxed{\text{False}}$: although the Jordan form of A^2 is conjugate to J^2, J^2 need not be in Jordan form.

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2. (a) Find a formula for the nth power of the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$.

- We diagonalize this matrix. The characteristic polynomial is $p(t) = t^2 - 5t - 6 = (t-6)(t+1)$ so the eigenvalues are $\lambda = 6, -1$.
- We can compute that $(1, 1)$ is a basis for the 6-eigenspace and $(5, -2)$ is a basis for the -1-eigenspace, so if we take $Q = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}$, $Q^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$, then $Q^{-1}AQ = D$.
- We then have $A^n = QD^nQ^{-1} = \boxed{\frac{1}{7}\begin{bmatrix} 2 \cdot 6^n + 5(-1)^n & 5 \cdot 6^n - 5(-1)^n \\ 2 \cdot 6^n - 2(-1)^n & 5 \cdot 6^n + 2 \cdot (-1)^n \end{bmatrix}}$.

(b) In Diagonalizistan there are two cities: City A and City B. Each year, 2/5 of the residents of City A move to City B, and 2/3 of the residents of City B move to City A; the remaining residents stay in their current city. If in year 0 the populations of Cities A and B are 2000 and 6000 residents respectively, find the populations of the two cities in year n and determine what happens as $n \to \infty$.

- Let a_n be the population in city A in year n and b_n be the population in city B in year n.
- Then the given information implies that $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 3/5 & 2/3 \\ 2/5 & 1/3 \end{bmatrix}^n \begin{bmatrix} 2000 \\ 6000 \end{bmatrix}$.
- To compute the matrix power, we diagonalize $A = \begin{bmatrix} 3/5 & 2/3 \\ 2/5 & 1/3 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = t^2 - \frac{14}{15}t - \frac{1}{15} = (t-1)(t+\frac{1}{15})$, so the eigenvalues are 1 and $-\frac{1}{15}$, with respective eigenspaces spanned by $\begin{bmatrix} 5/3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- So with $Q = \begin{bmatrix} 5/3 & -1 \\ 1 & 1 \end{bmatrix}$ we get $A^n = Q\begin{bmatrix} 1 & 0 \\ 0 & (-1/15)^n \end{bmatrix}Q^{-1} = \frac{1}{8}\begin{bmatrix} 5 + 3(-1/15)^n & 5 - 5(-1/15)^n \\ 3 - 3(-1/15)^n & 3 + 5(-1/15)^n \end{bmatrix}$.
- Then $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = A^n \begin{bmatrix} 2000 \\ 6000 \end{bmatrix} = \begin{bmatrix} 5000 - 3000(-1/15)^n \\ 3000 + 3000(-1/15)^n \end{bmatrix}$. Thus as $n \to \infty$, the populations approach 5000 in city A and 3000 in city B.

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3. For each matrix $M \in M_{n \times n}(\mathbb{C})$, find a basis for each of its generalized eigenspaces:

(a) $\begin{bmatrix} -4 & 9 \\ -4 & 8 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - M) = (t-2)^2$ so the eigenvalues are $\lambda = 2, 2$.
- We see that $(2I_2 - M)^2 = \begin{bmatrix} -4 & 9 \\ -4 & 8 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so every vector is a generalized 2-eigenvector: thus we can take any basis, such as $\boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$.

(b) $\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 2 \\ 2 & -2 & 0 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - M) = (t-1)(t-2)^2$ so the eigenvalues are $\lambda = 1, 2, 2$.
- First, $I_2 - M = \begin{bmatrix} 0 & -1 & -1 \\ 2 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix} \overset{RREF}{\to} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 1-eigenbasis $\boxed{\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}}$.
- $(2I_2 - M)^2 = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -2 & 2 & 2 \end{bmatrix} \overset{RREF}{\to} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 2-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$.

(c) $\begin{bmatrix} 1 & -1 & -5 \\ -1 & 2 & 8 \\ 1 & 0 & -2 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - M) = t^2(t-1)$ so the eigenvalues are $\lambda = 0, 0, 1$.
- First, $(-M)^2 = \begin{bmatrix} -3 & -3 & -3 \\ 5 & 5 & 5 \\ -1 & -1 & -1 \end{bmatrix} \overset{RREF}{\to} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 0-eigenbasis $\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$.
- Also, $I_2 - M = \begin{bmatrix} 0 & 1 & 5 \\ 1 & -1 & -8 \\ -1 & 0 & 3 \end{bmatrix} \overset{RREF}{\to} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 1-eigenbasis $\boxed{\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}}$.

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4. Suppose the characteristic polynomial of the 5×5 matrix A is $p(t) = t^3(t-1)^2$.

   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.

   - The eigenvalues are $t = \boxed{0, 0, 0, 1, 1}$. The possible dimensions of the 0-eigenspace are $\boxed{1, 2, 3}$ while the possible dimensions of the 1-eigenspace are $\boxed{1, 2}$.

   (b) Find the determinant and trace of A.

   - The determinant is the product of the eigenvalues (with multiplicity), which by (a) is $0^3 1^2 = \boxed{0}$, and the trace is the sum of the eigenvalues (with multiplicity), which by (a) is $3 \cdot 0 + 2 \cdot 1 = \boxed{2}$.

   (c) List all possible Jordan canonical forms of A up to equivalence.

   - For the 0-blocks, the possible sizes are 1-1-1, 2-1, or 3, and for the 1-blocks, the possible sizes are 1-1 or 2.
   - Thus, there are 6 possibilities up to equivalence:

$$\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

   (d) If $\ker(A)$ and $\ker(A - I)$ are both 2-dimensional, what is the Jordan canonical form of A?

   - If $\ker(A)$ is 2-dimensional then the 0-eigenspace has dimension 2, which means there are 2 Jordan blocks with eigenvalue 0, which therefore have sizes 2 and 1.
   - By the same logic, there are 2 Jordan blocks with eigenvalue 1, which therefore have sizes 1 and 1.
   - So the Jordan canonical form is $J = \boxed{\begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}$.

   (e) If A^3 is diagonalizable but A^2 is not, what is the Jordan canonical form of A?

   - The Jordan forms of A^2 and A^3 can be found from the square and cube of the Jordan form of A.
   - For the 1-blocks, since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, we see that if A has a 1-block of size 2, then A^3 would also have a 1-block of size 2 hence not be diagonalizable. So A must have two 1-blocks of size 1.
   - For the 0-blocks, since $\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ if A had three blocks of size 1 or blocks of sizes 1 and 2, then A^2 would be diagonalizable. However since we can see that $\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$, if the block had size 3 then A^2 would not be diagonalizable but A^3 would.

- Thus the only possible Jordan canonical form is $J = \boxed{\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}$.

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5. Find the Jordan canonical form of each matrix A over $\mathbb{C}$.

(a) $A = \begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^2$, so the eigenvalues are $\lambda = 0, 0$.
- We can calculate $\text{rank}(-A) = 1$ and $\text{rank}(-A)^2 = 0$.
- This means there is 1 Jordan block of size 2, so the Jordan canonical form is $\boxed{\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}}$.
- Alternatively, we could see this just by observing that the matrix is not diagonalizable, since then the only possible Jordan canonical form is the one listed above.

(b) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = (t-7)(t^2-7)$, so the eigenvalues are $\lambda = 7, -\sqrt{7}, \sqrt{7}$.
- Since the matrix is diagonalizable (either from the eigenvalue list, or because it is a real symmetric matrix), the Jordan form is the diagonalization $\boxed{\begin{bmatrix} 7 & & \\ & \sqrt{7} & \\ & & -\sqrt{7} \end{bmatrix}}$.

(c) $A = \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^2 - 12t + 37$, so the eigenvalues are $\lambda = 6 \pm i$.
- Since the eigenvalues are distinct, the matrix is diagonalizable, and the diagonalization $\boxed{\begin{bmatrix} 6+i & \\ & 6-i \end{bmatrix}}$ is also the Jordan canonical form.

(d) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = (t-1)(t-2)^2$, so the eigenvalues are $\lambda = 1, 2, 2$.
- Since 1 is a single eigenvalue, it must appear in a block of size 1.
- Also, we can calculate $\text{rank}(2I - A) = 2$ and $\text{rank}(2I - A)^2 = \text{rank}(2I - A)^3 = 1$.
- This means there is 1 Jordan 1-block of size 2, so the Jordan canonical form is $\boxed{\begin{bmatrix} 1 & & \\ & 2 & 1 \\ & & 2 \end{bmatrix}}$.
- Alternatively, we could see this just by observing that the matrix is not diagonalizable, since then the only possible Jordan canonical form is the one listed above.

(e) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^3 - 6t^2 + 12t - 8$, whose roots are $\lambda = 2, 2, 2$.
- This means all Jordan blocks have eigenvalue 2. To find the sizes, we calculate $\text{rank}(2I - A) = 2$, $\text{rank}(2I - A)^2 = 1$, $\text{rank}(2I - A)^3 = 0$. So there is only one Jordan block and it has size 3, so the Jordan canonical form is $\boxed{\begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}}$.

(f) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^3 - 6t^2 + 12t - 8$, whose roots are $\lambda = 2, 2, 2$.
- This means all Jordan blocks have eigenvalue 2. To find the sizes, we calculate $\text{rank}(2I - A) = 1$, $\text{rank}(2I - A)^2 = 0$.
- This means there are two Jordan blocks of sizes 1 and 2, so the Jordan canonical form is $\boxed{\begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix}}$.

(g) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = (t-1)^3(t-2)$ so the eigenvalues are $\lambda = 1, 1, 1, 2$.
- Since 2 is a single eigenvalue, it must occur in a single Jordan block of size 1.
- Also, we compute $\text{rank}(A - I) = 2$, $\text{rank}(A - I)^2 = \text{rank}(A - I)^3 = 1$. This means that for $\lambda = 1$, there is Jordan block of size 1 and one block of size 2.
- Hence the Jordan form is $\boxed{\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}}$.

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6. The goal of this problem is to give two proofs of Binet's formula for the Fibonacci numbers defined by the recurrence $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$; the next few terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Explicitly, for $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$, Binet's formula says that $F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$.

(a) Show that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and deduce that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- Note that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ by the recurrence.
- Thus, by a trivial induction, we see that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

(b) Find a formula for the nth power of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and use the result to deduce Binet's formula.

- We diagonalize this matrix. The characteristic polynomial is $p(t) = t^2 - t - 1$ so the eigenvalues are $\lambda = \frac{1 \pm \sqrt{5}}{2}$.
- Letting $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$, we can compute that $(\varphi, 1)$ is a basis for the φ-eigenspace and so $(\overline{\varphi}, 1)$ is a basis for the $\overline{\varphi}$-eigenspace, so if we take $Q = \begin{bmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{bmatrix}$, $Q^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{bmatrix}$, $D = \begin{bmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{bmatrix}$, then $A = QDQ^{-1}$.
- We then have $A^n = QD^nQ^{-1} = \begin{bmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & \overline{\varphi}^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & \varphi^{n-1} \\ -\overline{\varphi}^n & -\overline{\varphi}^{n-1} \end{bmatrix} =$ $\boxed{\frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - \overline{\varphi}^{n+1} & \varphi^n - \overline{\varphi}^n \\ \varphi^n - \overline{\varphi}^n & \varphi^{n-1} - \overline{\varphi}^{n-1} \end{bmatrix}}$.
- Then by (a), we see $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - \overline{\varphi}^{n+1} \\ \varphi^n - \overline{\varphi}^n \end{bmatrix}$, so $F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$ as claimed.

(c) Let W be the space of all real sequences $\{a_n\}_{n\geq 0}$ such that $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$. Show that W is a 2-dimensional vector space over $\mathbb{R}$.

- To show W is a vector space, simply verify the subspace criterion:
- [S1] W contains the zero sequence.
- [S2] If $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are in W, then for $c_n = a_n + b_n$ we have $c_{n+1} = a_{n+1} + b_{n+1} = (a_n + a_{n-1}) + (b_n + b_{n-1}) = c_n + c_{n-1}$ so $\{c_n\}_{n\geq 0}$ is in W.
- [S3] If $\{a_n\}_{n\geq 0}$ is in W, then for $d_n = ra_n$ we have $d_{n+1} = r(a_n + a_{n-1}) = d_n + d_{n-1}$ so $\{d_n\}_{n\geq 0}$ is in W.
- Furthermore, any such sequence is completely characterized by its 0th and 1st terms by the recurrence, and these values can be chosen freely. Thus, the map $T : W \to \mathbb{R}^2$ with $T(\{a_n\}_{n\geq 0}) = (a_0, a_1)$ is an isomorphism, and so W is 2-dimensional.

(d) With notation as in (c), show that the sequences $\{\varphi^n\}_{n\geq 0}$ and $\{\overline{\varphi}^n\}_{n\geq 0}$ are a basis for W. Deduce that there exist constants C and D such that $F_n = C\varphi^n + D\overline{\varphi}^n$ and then deduce Binet's formula.

- Note that φ and $\overline{\varphi}$ are the two roots of the quadratic $x^2 - x - 1 = 0$ as calculated in (a): thus, $\varphi^2 = \varphi + 1$ and $\overline{\varphi}^2 = \overline{\varphi} + 1$, so multiplying by φ^{n-1} and $\overline{\varphi}^{n-1}$ respectively yields $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$ and $\overline{\varphi}^{n+1} = \overline{\varphi}^n + \overline{\varphi}^{n-1}$.
- Thus, $\{\varphi^n\}_{n\geq 0}$ and $\{\overline{\varphi}^n\}_{n\geq 0}$ are both elements of W. Since they are linearly independent (since both sequences start with 1 at $n = 0$ but are different for $n = 1$) and W is 2-dimensional, they are a basis.
- Thus, there exist constants C and D such that $F_n = C\varphi^n + D\overline{\varphi}^n$.
- We can compute them by setting $n = 0, 1$ to see $0 = C + D$ and $1 = C\varphi + D\overline{\varphi}$, which yields $C = \dfrac{1}{\varphi - \overline{\varphi}} = \dfrac{1}{\sqrt{5}}$ and $D = -C = -\dfrac{1}{\sqrt{5}}$, again giving Binet's formula $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$.

Remark: Both of these methods extend generally to solve general linear recurrences of the form $a_{n+1} = C_1 a_n + C_2 a_{n-2} + \cdots + C_k a_{n-k}$ for constants $C_1, \ldots, C_k$. Additionally, the matrix formula in (a) is a good source of other Fibonacci identities.

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7. The goal of this problem is to find the Jordan form of the $n \times n$ "all 1s" matrix over an arbitrary field F. So let $n \geq 2$ and let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

(a) Show that the 0-eigenspace of A has dimension $n - 1$ and find a basis for it.

- Note that the 0-eigenspace is the same as the nullspace, which we can find by row-reducing.
- The reduced row-echelon form of A is clearly $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, which has $n-1$ non-pivot columns.
- Thus, the 0-eigenspace has dimension $n - 1$ and has a basis $(1, -1, 0, \ldots, 0)$, $(1, 0, -1, 0, \ldots, 0)$, ... , $(1, 0, 0, \ldots, 0, -1)$.

(b) If the characteristic of F does not divide n, find the remaining nonzero eigenvalue of A and a basis for the corresponding eigenspace, and show that A is diagonalizable. [Hint: Calculate the trace of A.]

- Since the characteristic polynomial has degree n, there must be exactly one additional eigenvalue.
- Since the trace of A is equal to n, we see that the sum of all the eigenvalues is n, so the other eigenvalue must be n.
- It is then quite easy to see that $(1, 1, \ldots, 1)$ is an n-eigenvector for A.
- There are thus two eigenspaces, the 0-eigenspace of dimension $n-1$ and the n-eigenspace of dimension 1. Since the sum of the eigenspace dimensions is n, A is diagonalizable.

(c) If the characteristic of F does divide n, show that A is not diagonalizable, and find its Jordan canonical form. [Hint: Note that char(F) dividing n is the same as saying that $n = 0$ in F.]

- By the same logic as in part (b), since the trace of A is equal to n, the other eigenvalue must be n.
- But now $n = 0$ in F, so in fact there is only one eigenvalue, namely, $\lambda = 0$.
- From the calculation in (a), the 0-eigenspace only has dimension $n-1$, so since it is now the only eigenspace, we conclude that A is not diagonalizable.
- For the Jordan form, we see that all eigenvalues are 0 and that the dimension of the 0-eigenspace is $n-1$. The only possibility, therefore, is that there are $n-2$ Jordan blocks of size 1 and 1 Jordan block of size 2, meaning the Jordan form is $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$.

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8. Let $A \in M_{n\times n}(\mathbb{C})$.

(a) Show that any Jordan-block matrix is similar to its transpose. [Hint: Reverse the Jordan basis.]

- Suppose J is the associated matrix $[T]_\beta^\beta$ for a linear transformation T with ordered basis $\beta = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n\}$: then $T\mathbf{v}_0 = \lambda\mathbf{v}_0$ and $T\mathbf{v}_k = \lambda\mathbf{v}_k + \mathbf{v}_{k-1}$.
- Therefore, with the ordered basis $\gamma = \{\mathbf{v}_n, \mathbf{v}_{n-1}, \dots, \mathbf{v}_0\}$, we see $[T]_\gamma^\gamma = \begin{bmatrix} \lambda & & & \\ 1 & \cdots & & \\ & \cdots & \lambda & \\ & & 1 & \lambda \end{bmatrix}$. Hence J is similar to its transpose. (Explicitly, $J^T = Q^{-1}JQ$, where Q is the "backwards diagonal" matrix.)

(b) If J is a matrix in Jordan canonical form, show that J is similar to its transpose.

- Suppose J is in Jordan canonical form with blocks $J_1, \dots, J_d$.
- By part (a) each of the Jordan blocks is similar to its transpose: say, with $J_i^T = Q_i^{-1}J_iQ_i$.
- Then for $Q = \begin{bmatrix} Q_1 & & \\ & \cdots & \\ & & Q_d \end{bmatrix}$ we see $Q^{-1}JQ = \begin{bmatrix} Q_1^{-1} & & \\ & \cdots & \\ & & Q_d^{-1} \end{bmatrix}\begin{bmatrix} J_1 & & \\ & \cdots & \\ & & J_d \end{bmatrix}\begin{bmatrix} Q_1 & & \\ & \cdots & \\ & & Q_d \end{bmatrix} = \begin{bmatrix} Q_1^{-1}J_1Q_1 & & \\ & \cdots & \\ & & Q_d^{-1}J_dQ_d \end{bmatrix} = \begin{bmatrix} J_1^T & & \\ & \cdots & \\ & & J_d^T \end{bmatrix} = J^T$. Thus, J is similar to its transpose.

(c) Show that A is similar to its transpose.

- By definition, A is similar to its Jordan canonical form J.
- By part (b), J is similar to J^T, and then by taking transposes, J^T is similar to A^T, since if $A = Q^{-1}JQ$ then $A^T = (Q^{-1}JQ)^T = Q^TJ^T(Q^{-1})^T = Q^TJ^T(Q^T)^{-1}$. Thus, A is similar to A^T.

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9. [Challenge] The goal of this problem is to characterize when the limit of matrix powers $\lim_{n\to\infty} A^n$ converges.

(a) Suppose $\lim_{n\to\infty} A^n = B$ exists. Show that every column of B lies in the 1-eigenspace of A. [Hint: Why is $AB = B$?]

- If $\lim_{n\to\infty} A^n = B$, multiplying by A yields $\lim_{n\to\infty} A^{n+1} = AB$. But the limit on the left is also B, by shifting the index of the variable n, so $B = AB$.
- If the ith column of B is $\mathbf{v}_i$, then since the ith column of AB is the matrix product of A with the ith column of B, we see that $A\mathbf{v}_i = \mathbf{v}_i$: thus, $\mathbf{v}_i$ is in the 1-eigenspace of A.

Now let J be a $d\times d$ Jordan block matrix with eigenvalue $\lambda \in \mathbb{C}$ and let $N = J - \lambda I_d$ be the matrix with 1s directly above the diagonal and 0s elsewhere.

(b) Show that $J^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{d}\lambda^{n-d}N^d$ for each $n \geq 1$.

- Note that $J^n = (\lambda I_d + N)^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{n}N^n$ by the binomial theorem and the fact that $NI_n = I_nN$.
- However, N^d is the zero matrix, as $N(\mathbf{e}_i) = \mathbf{e}_{i+1}$ where $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the standard basis and we view $\mathbf{e}_k = \mathbf{0}$ for $k > d$. Then $N^d(\mathbf{e}_i) = \mathbf{e}_{i+d} = \mathbf{0}$ for all i, so N^d is zero on all vectors.
- Thus, the terms past N^d are all zero, so we may ignore them. Thus we get $J^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{d}\lambda^{n-d}N^d$ as claimed.

(c) Show that $\lim_{n\to\infty} J^n$ exists if and only if $|\lambda| < 1$ or if $\lambda = 1$ and $d = 1$.

- From (a) we see that $J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \cdots & \binom{n}{d-1}\lambda^{n-d+1} \\ & \lambda^n & \cdots & \cdots \\ & & \cdots & n\lambda^{n-1} \\ & & & \lambda^n \end{bmatrix}$.
- Clearly $J^n = [1]$ if $\lambda = 1$ and $d = 1$ so J^n converges in that case, and if $|\lambda| < 1$ then every entry in J^n converges to zero so J^n converges as well.
- Conversely, in order for $\lim_{n\to\infty} J^n$ to exist, we require λ^n to converge as $n \to \infty$. This clearly requires $|\lambda| \le 1$ since otherwise $|\lambda^n| \to \infty$, and if $|\lambda| = 1$ then if $\lambda = e^{i\theta}$ then $\lambda^n = e^{in\theta}$, which does not converge as $n \to \infty$ unless $\lambda = 1$. Furthermore, if $\lambda = 1$ and $d > 1$, then the entries immediately above the diagonal in J^n are equal to n, which does not converge.
- Therefore, if $\lim_{n\to\infty} J^n$ exists, then we must have $|\lambda| < 1$ or $\lambda = 1$ and $d = 1$, as claimed.

(d) Let A be a square complex matrix. Show that $\lim_{n\to\infty} A^n$ exists if and only if 1 is the only eigenvalue of A of absolute value ≥ 1 and the dimension of the 1-eigenspace equals its multiplicity as a root of the characteristic polynomial.

- If J is the Jordan canonical form of A with $J = PAP^{-1}$, then $J^n = PA^nP^{-1}$ and $A^n = P^{-1}J^nP$, so $\lim_{n\to\infty} A^n$ exists if and only if $\lim_{n\to\infty} J^n$ exists.
- Since J is block-diagonal, $\lim_{n\to\infty} J^n$ exists if and only if the limit of the nth power of each Jordan block in J exists. But by part (b), this is the case if and only if each Jordan block either has $|\lambda| < 1$, or if $\lambda = 1$ and $d = 1$.
- Equivalently, this means that the only eigenvalue of absolute value ≥ 1 is $\lambda = 1$, and if each Jordan block with $\lambda = 1$ has size 1. This is equivalent to saying that every generalized 1-eigenvector is a 1-eigenvector, which is in turn equivalent to saying that the dimension of the 1-eigenspace equals its multiplicity as a root of the characteristic polynomial, as claimed.

(e) Suppose M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1) such that some power of M has all positive entries. Show that $\lim_{n\to\infty} M^n$ converges to a matrix whose columns are all 1-eigenvectors of M. [Hint: Use the results of the challenge problem from homework 9 applied to an appropriate power of M.]

- Suppose M^n has all positive entries. Then M^n is still a stochastic matrix, so by the challenge problem from homework 9, we know that all eigenvalues of M^n satisfy $|\lambda| < 1$ or $\lambda = 1$, and the 1-eigenspace has dimension 1.
- By the spectral mapping theorem, if μ is an eigenvalue of M then μ^n is an eigenvalue of M^n, so either $|\mu^n| < 1$ or $\mu^n = 1$, and the total dimension of all the eigenspaces with $\mu^n = 1$ is 1.
- But since 1 is an eigenvalue of M as well, it must be the only eigenvalue with $\mu^n = 1$, and then the remaining eigenvalues have $|\mu| < 1$. Hence by (c), the matrix limit $\lim_{n\to\infty} M^n$ converges.
- Specifically, if J is the Jordan canonical form of M with $M = P^{-1}JP$, where the (1,1)-entry of J is 1 and the first column of P is a 1-eigenvector of M, then $\lim_{n\to\infty} J^n$ is the matrix with (1,1)-entry equal to 1 and other entries equal to zero. Then the product $\lim_{n\to\infty} M^n = P^{-1}[\lim_{n\to\infty} J^n]P$ has all columns proportional to the first column of P, as claimed. Finally, by (a), since $\lim_{n\to\infty} M^n$ converges, all of the columns are 1-eigenvectors of M.
