E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 9, due Fri Mar 27th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a finite-dimensional vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) If $\dim(V) = n$ and T has n distinct eigenvalues in F, then T is diagonalizable.

   (b) If $\dim(V) = n$ and T is diagonalizable, then T has n distinct eigenvalues in F.

   (c) If A is a diagonalizable $n \times n$ matrix, then so is $A + I_n$.

   (d) For any scalar λ, the λ-eigenspace of T is a subspace of the generalized λ-eigenspace of T.

   (e) For any λ, a chain of generalized λ-eigenvectors is linearly independent.

   (f) There always exists a basis β of V consisting of generalized eigenvectors of T.

   (g) If all eigenvalues of T lie in F, then there exists a basis β of V of generalized eigenvectors for T.

   (h) There always exists some basis β of V such that the matrix $[T]_\beta^\beta$ is in Jordan canonical form.

   (i) Every matrix $A \in M_{n\times n}(\mathbb{C})$ has a Jordan canonical form.

   (j) If a matrix is diagonalizable, then its Jordan canonical form is diagonal.

   (k) If the Jordan canonical form of a matrix is diagonal, then the matrix is diagonalizable.

   (l) Two matrices are similar if and only if they have equivalent Jordan canonical forms.

   (m) If J is the Jordan canonical form of A, then $J + I_n$ is the Jordan canonical form of $A + I_n$.

   (n) If J is the Jordan canonical form of A, then J^2 is the Jordan canonical form of A^2.

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2. (a) Find a formula for the nth power of the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$.

   (b) In Diagonalizistan there are two cities: City A and City B. Each year, 2/5 of the residents of City A move to City B, and 2/3 of the residents of City B move to City A; the remaining residents stay in their current city. If in year 0 the populations of Cities A and B are 2000 and 6000 residents respectively, find the populations of the two cities in year n and determine what happens as $n \to \infty$.

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3. For each matrix $M \in M_{n\times n}(\mathbb{C})$, find a basis for each of its generalized eigenspaces:

   (a) $\begin{bmatrix} -4 & 9 \\ -4 & 8 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 2 \\ 2 & -2 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & -1 & -5 \\ -1 & 2 & 8 \\ 1 & 0 & -2 \end{bmatrix}$.

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4. Suppose the characteristic polynomial of the 5×5 matrix A is $p(t) = t^3(t-1)^2$.

   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.

   (b) Find the determinant and trace of A.

   (c) List all possible Jordan canonical forms of A up to equivalence.

   (d) If $\ker(A)$ and $\ker(A - I)$ are both 2-dimensional, what is the Jordan canonical form of A?

   (e) If A^3 is diagonalizable but A^2 is not, what is the Jordan canonical form of A?

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5. Find the Jordan canonical form of each matrix A over $\mathbb{C}$.

(a) $A = \begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$. (c) $A = \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$. (d) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix}$. (f) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix}$. (g) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

6. The goal of this problem is to give two proofs of Binet's formula for the Fibonacci numbers defined by the recurrence $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$; the next few terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Explicitly, for $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1-\sqrt{5}}{2}$, Binet's formula says that $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$.

   (a) Show that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and deduce that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

   (b) Find a formula for the nth power of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and use the result to deduce Binet's formula.

   (c) Let W be the space of all real sequences $\{a_n\}_{n\geq 0}$ such that $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$. Show that W is a 2-dimensional vector space over $\mathbb{R}$.

   (d) With notation as in (c), show that the sequences $\{\varphi^n\}_{n\geq 0}$ and $\{\overline{\varphi}^n\}_{n\geq 0}$ are a basis for W. Deduce that there exist constants C and D such that $F_n = C\varphi^n + D\overline{\varphi}^n$ and then deduce Binet's formula.

   Remark: Both of these methods extend generally to solve general linear recurrences of the form $a_{n+1} = C_1 a_n + C_2 a_{n-2} + \cdots + C_k a_{n-k}$ for constants $C_1, \ldots, C_k$. Additionally, the matrix formula in (a) is a good source of other Fibonacci identities.

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7. The goal of this problem is to find the Jordan form of the $n \times n$ "all 1s" matrix over an arbitrary field F. So let $n \geq 2$ and let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

   (a) Show that the 0-eigenspace of A has dimension $n-1$ and find a basis for it.

   (b) If the characteristic of F does not divide n, find the remaining nonzero eigenvalue of A and a basis for the corresponding eigenspace, and show that A is diagonalizable. [Hint: Calculate the trace of A.]

   (c) If the characteristic of F does divide n, show that A is not diagonalizable, and find its Jordan canonical form. [Hint: Note that char(F) dividing n is the same as saying that $n = 0$ in F.]

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8. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) Show that any Jordan-block matrix is similar to its transpose. [Hint: Reverse the Jordan basis.]

   (b) If J is a matrix in Jordan canonical form, show that J is similar to its transpose.

   (c) Show that A is similar to its transpose.

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9. [Challenge] The goal of this problem is to characterize when the limit of matrix powers $\lim_{n\to\infty} A^n$ converges.

   (a) Suppose $\lim_{n\to\infty} A^n = B$ exists. Show that every column of B lies in the 1-eigenspace of A. [Hint: Why is $AB = B$?]

   Now let J be a $d \times d$ Jordan block matrix with eigenvalue $\lambda \in \mathbb{C}$ and let $N = J - \lambda I_d$ be the matrix with 1s directly above the diagonal and 0s elsewhere.

   (b) Show that $J^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{d}\lambda^{n-d}N^d$ for each $n \geq 1$.

   (c) Show that $\lim_{n\to\infty} J^n$ exists if and only if $|\lambda| < 1$ or if $\lambda = 1$ and $d = 1$.

   (d) Let A be a square complex matrix. Show that $\lim_{n\to\infty} A^n$ exists if and only if 1 is the only eigenvalue of A of absolute value ≥ 1 and the dimension of the 1-eigenspace equals its multiplicity as a root of the characteristic polynomial.

   (e) Suppose M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1) such that some power of M has all positive entries. Show that $\lim_{n\to\infty} M^n$ converges to a matrix whose columns are all 1-eigenvectors of M. [Hint: Use the results of the challenge problem from homework 8 applied to an appropriate power of M.]
