E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 8 Solutions

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1. Let V be a vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) If $T(\mathbf{v}) = \lambda\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of T.

   - $\boxed{\text{False}}$: we would need to exclude $\mathbf{v} = \mathbf{0}$ here, since by definition $\mathbf{v} = \mathbf{0}$ is not an eigenvector.

   (b) Every linear transformation on V has at least one eigenvector.

   - $\boxed{\text{False}}$: there are linear transformations with no eigenvectors, e.g., integration on $\mathbb{R}[x]$.

   (c) If V is finite-dimensional, every linear transformation on V has at least one eigenvector.

   - $\boxed{\text{False}}$: the characteristic polynomial may have no roots in the scalar field. For example, the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (y, -x)$ has no real eigenvalues hence no eigenvectors.

   (d) Any two eigenvectors of T are linearly independent.

   - $\boxed{\text{False}}$: this is only true if the associated eigenvalues are different.

   (e) The sum of two eigenvectors of T is also an eigenvector of T.

   - $\boxed{\text{False}}$: usually not, e.g, for $T(x, y) = (x, 2y)$ then $(1, 0)$ and $(0, 1)$ are eigenvectors, but $(1, 1)$ is not.

   (f) The sum of two eigenvalues of T is also an eigenvalue of T.

   - $\boxed{\text{False}}$: usually not, e.g., for $T(x, y) = (x, 2y)$ then 1 and 2 are eigenvalues, but 3 is not.

   (g) If two matrices are similar, then they have the same eigenvectors.

   - $\boxed{\text{False}}$: as we showed, if $A = QBQ^{-1}$ and $A\mathbf{v} = \lambda\mathbf{v}$ then $B(Q\mathbf{v}) = \lambda(Q\mathbf{v})$. This means the eigenspaces of A and B are related by left-multiplication by Q, but are not necessarily equal.

   (h) If two matrices have the same eigenvalues, then they are similar.

   - $\boxed{\text{False}}$: for example, the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same eigenvalues but are not similar (the latter is not diagonalizable but the former is).

   (i) If two matrices are similar, then they have the same eigenvalues.

   - $\boxed{\text{True}}$: similar matrices have the same characteristic polynomial hence the same eigenvalues.

   (j) If $\dim(V) = n$, then T has at most n distinct eigenvalues in F.

   - $\boxed{\text{True}}$: the characteristic polynomial $p(t) = \det(tI - A)$ has degree n hence at most n roots in F.

   (k) If $\dim(V) = n$, then T has exactly n distinct eigenvalues in F.

   - $\boxed{\text{False}}$: the characteristic polynomial may have repeated roots, in which case it would have fewer than n distinct roots. It may also have irreducible terms of degree > 1, which would further lower the number of roots.

   (l) If the characteristic polynomial of A is $p(t) = t(t - 1)^2$, then the 1-eigenspace of A has dimension 2.

   - $\boxed{\text{False}}$: although 1 is a double root of the characteristic polynomial, this means only that the 1-eigenspace can have dimension 1 or 2.

   (m) If the characteristic polynomial of A is $p(t) = t(t - 1)^2$, then the only vector $\mathbf{v}$ with $A\mathbf{v} = 3\mathbf{v}$ is $\mathbf{v} = \mathbf{0}$.

   - $\boxed{\text{True}}$: such a vector would be an element of the 3-eigenspace, but since 3 is not an eigenvalue of A, the 3-eigenspace is trivial.

   (n) V has a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T if and only if T is diagonalizable.

   - $\boxed{\text{True}}$: $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$ if and only if $[T]_\beta^\beta$ is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$.

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2. For each matrix A over each field F, (i) find all eigenvalues of A over F, (ii) find a basis for each eigenspace of A, and (iii) determine whether or not A is diagonalizable over F and if so find an invertible matrix Q and diagonal matrix D such that $D = Q^{-1}AQ$.

(a) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{R}$.

- The characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-3 & -1 \\ 2 & t-5 \end{vmatrix} = t^2 - 8t + 17$.
- The roots of this polynomial are $\lambda = 4 \pm i$. Thus, there are $\boxed{\text{no eigenvalues over } \mathbb{R}}$, and so it is $\boxed{\text{not diagonalizable}}$.

(b) The matrix $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Q}$.

- The characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-1 & -1 & 1 \\ 2 & t-3 & 2 \\ 1 & 0 & t-1 \end{vmatrix} = (t-1)(t-2)^2$.
- Thus, the eigenvalues are $\lambda = \boxed{1, 2, 2}$.
- Row-reducing $\lambda I - A$ yields the 1-eigenspace basis $\boxed{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$, and the 2-eigenspace basis $\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$.
- Since the 2-eigenspace is only 1-dimensional, the matrix is $\boxed{\text{not diagonalizable}}$.

(c) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{C}$.

- The characteristic polynomial was calculated in (a) as $p(t) = t^2 - 8t + 17$.
- Over $\mathbb{C}$, the eigenvalues are $\lambda = \boxed{4+i,\ 4-i}$ with respective eigenbases $\boxed{\begin{bmatrix} 1-i \\ 2 \end{bmatrix}}$ and $\boxed{\begin{bmatrix} 1+i \\ 2 \end{bmatrix}}$.
- The matrix $\boxed{\text{is diagonalizable}}$: we can take $D = \begin{bmatrix} 4+i & 0 \\ 0 & 4-i \end{bmatrix}$ and $Q = \begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}$.

(d) The matrix $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$ over $\mathbb{C}$.

- The characteristic polynomial is $\det(tI - A) = (t-1)(t-2)^2$. Thus, the eigenvalues are $\lambda = \boxed{1, 2, 2}$.
- Row-reducing $\lambda I - A$ yields the 1-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$ and the 2-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}$.
- Since the sum of the eigenspace dimensions is 3, the matrix $\boxed{\text{is diagonalizable}}$: the diagonalization is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ via the matrix $Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

(e) The matrix $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$ over $\mathbb{R}$.

- The characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t+5 & -9 \\ 4 & t-7 \end{vmatrix} = t^2 - 2t + 1$ with roots $\lambda = \boxed{1, 1}$.
- Row-reducing $\lambda I - A = I - A = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$ yields $\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$, with nullspace basis $\boxed{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$.
- Since the 1-eigenspace is only 1-dimensional, the matrix is $\boxed{\text{not diagonalizable}}$.

(f) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ over $\mathbb{C}$.

- The characteristic polynomial is $\det(tI - A) = (t-6)(t^2-3)$ with roots $\lambda = \boxed{6, \sqrt{3}, -\sqrt{3}}$.
- Row-reducing $\lambda I - A$ for each of the three possible eigenvalues λ yields that each eigenspace is 1-dimensional.
- Explicitly: 6-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$, $\sqrt{3}$-eigenbasis $\boxed{\begin{bmatrix} -1-\sqrt{3} \\ -1+\sqrt{3} \\ 2 \end{bmatrix}}$, $-\sqrt{3}$-eigenbasis $\boxed{\begin{bmatrix} -1+\sqrt{3} \\ -1-\sqrt{3} \\ 2 \end{bmatrix}}$.
- Since the sum of the eigenspace dimensions is 3, the matrix $\boxed{\text{is diagonalizable}}$.
- The diagonalization is $\begin{bmatrix} 6 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \end{bmatrix}$ via the matrix $Q = \begin{bmatrix} 1 & -1-\sqrt{3} & -1+\sqrt{3} \\ 1 & -1+\sqrt{3} & -1-\sqrt{3} \\ 1 & 2 & 2 \end{bmatrix}$.

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3. For each operator $T : V \to V$ on each vector space V, (i) find all its eigenvalues and a basis for each eigenspace, and (ii) determine whether the operator is diagonalizable and if so, find a basis for which $[T]_\beta^\beta$ is diagonal:

(a) The map $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $T(x, y) = (x + 4y, 3x + 5y)$.

- With respect to the standard basis β the associated matrix is $A = [T]_\beta^\beta = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = \det(tI - A) = t^2 - 6t - 1 = (t-7)(t+1)$. Hence the eigenvalues are $\lambda = \boxed{7, -1}$, and row-reducing yields corresponding eigenbases $\boxed{(2,3)}$ and $\boxed{(-2,1)}$.
- Since the sum of the eigenspace dimensions is 2, the transformation $\boxed{\text{is diagonalizable}}$: the diagonalization is $\begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$ via the basis $\beta = \boxed{\{(2,3), (-2,1)\}}$.

(b) The derivative operator $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $D(p) = p'$.

- With respect to the standard basis $\beta = \{1, x, x^2\}$, the associated matrix is $[D]_\beta^\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.
- Since this matrix is upper-triangular, the eigenvalues are just the diagonal elements $\lambda = \boxed{0, 0, 0}$. Row-reducing yields that the 0-eigenspace is 1-dimensional and has basis $\boxed{\{1\}}$.
- Since the eigenspace is only 1-dimensional, the transformation is $\boxed{\text{not diagonalizable}}$.

(c) The transpose map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(M) = M^T$.

- With standard basis $\beta = \{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$, the associated matrix is $A = [T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = \det(tI - A) = (t-1)^3(t+1)$. Hence the eigenvalues are $\lambda = \boxed{1, 1, 1, -1}$.
- Row-reducing yields corresponding eigenbases $\boxed{\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}}$ and $\boxed{\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}}$. Alternatively, one could observe that the 1-eigenspace is the space of symmetric matrices, while the (-1)-eigenspace is the space of skew-symmetric matrices.
- Since the sum of the eigenspace dimensions is 4, the transformation $\boxed{\text{is diagonalizable}}$: the diagonalization is $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$ via the basis $\beta = \boxed{\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}}$.

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4. Let F be a field and let L and R be the left shift and right shift operators on infinite sequences of elements of F, defined by $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots)$ and $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

   (a) Find all of the eigenvalues and a basis for each eigenspace of L.

   - Solving $L(a_1, a_2, a_3, a_4, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ gives $(a_2, a_3, a_4, a_5, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ whence $a_2 = \lambda a_1$, $a_3 = \lambda a_2$, ... , $a_{i+1} = \lambda a_i$, ...
   - It is then easy to see that the λ-eigenspace is 1-dimensional and spanned by the vector $(1, \lambda, \lambda^2, \lambda^3, \dots)$. In particular, every element $\lambda \in F$ is an eigenvalue of L.

   (b) Find all of the eigenvalues and a basis for each eigenspace of R.

   - Solving $R(a_1, a_2, a_3, a_4, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ gives $(0, a_1, a_2, a_3, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ whence $\lambda a_1 = 0$, $\lambda a_2 = a_1$, ... , $\lambda a_{i+1} = a_i$.
   - If $\lambda \neq 0$ then cancelling λ gives $a_i = 0$ for all i (but the zero vector is not an eigenvector by definition), while if $\lambda = 0$ then again we see $a_i = 0$ for all i.
   - Thus R has no eigenvalues since $(a_1, a_2, a_3, \dots) = \lambda(0, a_1, a_2, a_3, \dots)$ forces $a_1 = a_2 = a_3 = \dots = 0$.

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5. Suppose V is a vector space and $S, T : V \to V$ are linear operators on V.

   (a) If S and T commute (i.e., $ST = TS$), show that S maps each eigenspace of T into itself.

   - Suppose $T\mathbf{v} = \lambda \mathbf{v}$. Then $\lambda(S\mathbf{v}) = S(\lambda \mathbf{v}) = S(T\mathbf{v}) = T(S\mathbf{v})$ so $S\mathbf{v}$ is also a λ-eigenvector of T.

   (b) If $\mathbf{v}$ is an eigenvector of T, show that it is also an eigenvector of T^n for any positive integer n.

   - Suppose $T\mathbf{v} = \lambda \mathbf{v}$. Then $T^2\mathbf{v} = T(T\mathbf{v}) = T(\lambda \mathbf{v}) = \lambda(T\mathbf{v}) = \lambda^2 \mathbf{v}$.
   - By repeating this argument (equivalently, by a trivial induction) we see that $T^n\mathbf{v} = \lambda^n \mathbf{v}$, so $\mathbf{v}$ is also an eigenvector of T^n with corresponding eigenvalue λ^n.

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6. Suppose A is an invertible $n \times n$ matrix and that $p(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0$ is its characteristic polynomial. Note that $a_0 = (-1)^n \det(A)$ is nonzero.

   (a) If $B = -\dfrac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \dots + a_2 A + a_1 I_n)$, show that $AB = I_n$. [Hint: Cayley-Hamilton.]

   - By the Cayley-Hamilton theorem, we know that $A^n + a_{n-1}A^{n-1} + \dots + a_2 A^2 + a_1 A + a_0 I_n = 0$, so rearranging yields $A^n + a_{n-1}A^{n-1} + \dots + a_2 A^2 + a_1 A = -a_0 I_n$.
   - From the definition of B, we can multiply through by A to see that $AB = -\dfrac{1}{a_0}(A^n + a_{n-1}A^{n-1} + \dots + a_2 A^2 + a_1 A) = -\dfrac{1}{a_0}(-a_0 I_n) = I_n$, as claimed.

   (b) Show that there exists a polynomial $q(x)$ of degree at most $n-1$ such that $A^{-1} = q(A)$.

   - By part (a) we see that $AB = I_n$ so that $B = A^{-1}$.
   - The desired statement is then immediate from the expression in part (a), since the expression for B is a polynomial in A of degree $n-1$.

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7. Suppose V is finite-dimensional and $T : V \to V$ is a projection, so that $T^2 = T$.

   (a) Show that the only possible eigenvalues of T are 0 and 1.

   - Suppose $\mathbf{v}$ is an eigenvector of T with eigenvalue λ, so that $T(\mathbf{v}) = \lambda \mathbf{v}$.
   - Then $\lambda^2 \mathbf{v} = T^2(\mathbf{v}) = T(\mathbf{v}) = \lambda \mathbf{v}$, so $(\lambda^2 - \lambda)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$ this means $\lambda^2 = \lambda$, so $\lambda = 0, 1$.

   (b) Show that T is diagonalizable. [Hint: See homework 5.]

   - As shown in problem 7b of homework 5, if β is a basis for $\ker(T)$ followed by a basis for $\operatorname{im}(T)$, then $[T]_\beta^\beta$ is diagonal with diagonal entries all 0s (for the kernel elements) and 1s (for the image elements). This β is a diagonalizing basis for T.

(c) Suppose A and B are projection maps on V of the same rank. Show that A and B are similar. Deduce that up to similarity, there are $\dim(V)+1$ different projection maps on V.

- From (a) and (b) together, we can characterize a projection map up to similarity by its diagonalization, whose diagonal entries must be either 0s or 1s.
- If $\dim(V)=n$, there are clearly $n+1$ such matrices (with n zero entries, $n-1$ zero entries, ... , 1 zero entry, 0 zero entries) and each of these matrices has a different rank: namely, 0, 1, ... , $n-1$, n. Thus, up to similarity, there are $n+1$ different projection maps on V.

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8. The goal of this problem is to give some counterexamples for results about orthogonal complements, projections, best approximations, and adjoints in infinite-dimensional spaces. Let V be the vector space of infinite real sequences $\{a_i\}_{i\geq 1}=(a_1,a_2,\dots)$ with only finitely many nonzero terms, with inner product given by $\langle\{a_i\},\{b_i\}\rangle=\sum_{i=1}^{\infty}a_ib_i$. (Note that this sum converges since only finitely many terms are nonzero.) Let $\mathbf{e}_i$ be the ith unit coordinate vector and observe that $\{\mathbf{e}_i\}_{i\geq 1}$ is an orthonormal basis for V. Now for each $n\geq 2$, let $\mathbf{v}_n=\mathbf{e}_1-\mathbf{e}_n$ and define $W=\operatorname{span}(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\dots)$.

(a) Show that $\mathbf{e}_1\notin W$ so that W is a proper subspace of V, but that $W^{\perp}=\{\mathbf{0}\}$.

- If we have $\mathbf{e}_1=a_2\mathbf{v}_2+a_3\mathbf{v}_3+\cdots+a_n\mathbf{v}_n$, then expanding out yields $\mathbf{e}_1=(a_1+\cdots+a_n)\mathbf{e}_1-a_2\mathbf{e}_2-a_3\mathbf{e}_3-\cdots-a_n\mathbf{e}_n$ so since the $\mathbf{e}_i$ are linearly independent, we would have $a_1+\cdots+a_n=1$ and also $a_2=a_3=\cdots=a_n=0$, but this is contradictory. Thus $\mathbf{e}_1\notin W$.
- Next, if $\mathbf{w}=b_1\mathbf{e}_1+\cdots+b_n\mathbf{e}_n$ is an element of $W^{\perp}$, we have $0=\langle b_1\mathbf{e}_1+\cdots+b_n\mathbf{e}_n,\mathbf{v}_k\rangle=b_1-b_k$. Thus $b_k=b_1$ for all k, but since only finitely many b_k can be nonzero, we must have $b_1=b_2=\cdots=0$, and so $\mathbf{w}=\mathbf{0}$.

(b) Show that $W^{\perp}+W\neq V$ and that $(W^{\perp})^{\perp}\neq W$.

- By (a) we have $W^{\perp}+W=W\neq V$, and also $(W^{\perp})^{\perp}=(\{\mathbf{0}\})^{\perp}=V\neq W$.

(c) For any $\mathbf{v}\notin W$, show that there does not exist any choice of $\mathbf{w}\in W$ and $\mathbf{w}^{\perp}\in W^{\perp}$ such that $\mathbf{v}=\mathbf{w}+\mathbf{w}^{\perp}$. Conclude that there is not a well-defined orthogonal projection map of V onto W.

- Suppose we had such $\mathbf{w},\mathbf{w}^{\perp}$. From (a) we know that $W^{\perp}=\{\mathbf{0}\}$, so the only possible choice would be $\mathbf{w}^{\perp}=\mathbf{0}$. But this would imply $\mathbf{v}=\mathbf{w}$ which is impossible since $\mathbf{v}$ is not in W.

(d) Show that there exists $\mathbf{w}_n\in W$ such that $||\mathbf{w}_n-\mathbf{e}_1||=1/n$ for any positive integer n. Deduce there is no possible best approximation $\mathbf{w}\in W$ to $\mathbf{e}_1$, namely with $||\mathbf{w}-\mathbf{e}_1||\leq||\mathbf{w}'-\mathbf{e}_1||$ for all $\mathbf{w}'\in W$. [Hint: Take $\mathbf{w}_n=(1,-1/n^2,-1/n^2,\dots,-1/n^2,0,0,\dots)$.]

- Consider the vector $\mathbf{w}_n=(1,-1/n^2,-1/n^2,\dots,-1/n^2,0,0,\dots)$ with first entry 1 followed by n^2 entries equal to $-1/n^2$, and other entries 0.
- Then $||\mathbf{w}_n-\mathbf{e}_1||^2=\left|\left|(0,-1/n^2,-1/n^2,\dots,-1/n^2,0,0,\dots)\right|\right|^2=n^2\cdot(1/n^2)^2=1/n^2$ so $||\mathbf{w}-\mathbf{e}_1||=1/n$ as desired.
- Since $1/n\to 0$ as $n\to\infty$, a best approximation vector $\mathbf{w}$ would necessarily have $||\mathbf{w}-\mathbf{e}_1||=0$, but this is impossible since $\mathbf{e}_1\notin W$.

(e) Let $T:V\to V$ be the linear transformation defined by setting $T(\mathbf{e}_n)=\sum_{i=1}^{n}\mathbf{e}_i$ for each $i\geq 1$. If T had an adjoint $T^*:V\to V$, show that infinitely many components of $T^*(\mathbf{e}_1)$ would be nonzero. Deduce that T^* cannot exist.

- By hypothesis we have $\langle\mathbf{e}_k,T^*(\mathbf{e}_1)\rangle=\langle T(\mathbf{e}_k),\mathbf{e}_1\rangle=\langle\sum_{i=1}^{n}\mathbf{e}_i,\mathbf{e}_1\rangle=1$ for each $k\geq 1$.
- Conjugating then yields $\langle T^*(\mathbf{e}_1),\mathbf{e}_k\rangle=1$: but this inner product is the coefficient of $\mathbf{e}_k$ in $T^*(\mathbf{e}_1)$, so we would necessarily have $T^*(\mathbf{e}_1)=\sum_{k=1}^{\infty}\mathbf{e}_k$. But this vector is not an element of V since it has infinitely many nonzero components. This is a contradiction, so T^* cannot exist.

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9. [Challenge] The goal of this problem is to prove various results about eigenvalues of complex matrices and stochastic matrices. Let $A \in M_{n\times n}(\mathbb{C})$, define $\rho_i(A)$ to be the sum of the absolute values of the entries in the ith row of A, and define $\rho(A) = \max_{1\leq i\leq n} \rho_i(A)$.

   (a) Define the ith Gershgorin disk C_i to be the disc in $\mathbb{C}$ centered at $a_{i,i}$ with radius $r_i(A) = \rho_i(A) - |a_{i,i}|$. Prove Gershgorin's disc theorem: every eigenvalue of A is contained in one of the Gershgorin disks of A. [Hint: If $\mathbf{v} = (x_1, \dots, x_n)$ is an eigenvector where x_k has the largest absolute value among the entries of $\mathbf{v}$, show that $|\lambda x_k - a_{k,k}x_k| \leq r_i(A)\,|x_k|$ by noting that λx_k is the kth component of $A\mathbf{v}$.]

   - Suppose $\mathbf{v} = (x_1, \dots, x_n)$ is an eigenvector with eigenvalue λ: then by taking the kth component of $A\mathbf{v} = \lambda\mathbf{v}$ we see that $\sum_{j=1}^{n} a_{k,j}x_j = \lambda x_k$.
   - Thus, $|\lambda x_k - a_{k,k}x_k| = \left|\sum_{j=1}^{n} a_{k,j}x_j - a_{k,k}x_j\right| = \left|\sum_{j\neq k} a_{k,j}x_j\right| \leq \sum_{j\neq k} |a_{k,j}|\,|x_j| \leq \sum_{j\neq k} |a_{k,j}|\,|x_k| = (\rho_i(A) - |a_{k,k}|)\,|x_k| = r_i(A)\,|x_k|$, where we used the triangle inequality at the first $\leq$ and the fact that $|x_j| \leq |x_k|$ for each j in the second $\leq$.
   - Since $|x_k| > 0$ because $\mathbf{v} \neq \mathbf{0}$, dividing through by $|x_k|$ yields $|\lambda - a_{k,k}| \leq r_i(A)$: in other words, λ lies within a Gershgorin disk of A. This holds for all eigenvalues λ, so all eigenvalues lie within Gershgorin disks of A.

   (b) For any eigenvalue λ of $A \in M_{n\times n}(\mathbb{C})$, prove that $|\lambda| \leq \rho(A)$.

   - By Gershgorin's disc theorem from (a), we have $|\lambda - a_{k,k}| \leq \rho_k(A) - |a_{k,k}|$ for some k.
   - Then by the triangle inequality, we have $|\lambda| \leq |\lambda - a_{k,k}| + |a_{k,k}| = \rho_k(A)$. Since the maximum of a set is greater than or equal to all of the elements, this immediately yields $|\lambda| \leq \max_{1\leq i\leq n} \rho_i(A)$.

   (c) Prove that if $A \in M_{n\times n}(\mathbb{R})$ has positive entries and there exists an eigenvalue λ such that $|\lambda| = \rho(A)$, then $\lambda = \rho(A)$ and the λ-eigenspace is 1-dimensional and spanned by the vector $\mathbf{v} = (1, 1, \dots, 1)$. [Hint: Analyze when equality can hold in (a) and (b).]

   - If $|\lambda| = \rho(A)$ in part (b), then we must have equality in the triangle inequality: $|\lambda| = |\lambda - a_{k,k}| + |a_{k,k}|$. This occurs if and only if λ is real and $\lambda \geq a_{k,k}$, so since $a_{k,k}$ this means λ is a positive real number. Since $\rho(A)$ is also a positive real number, this means $\lambda = \rho(A)$.
   - Furthermore, if $\mathbf{v} = (x_1, x_2, \dots, x_k)$ is a corresponding eigenvector (which is necessarily real, since λ is real), then to get equality in the argument for (a), we must have $\left|\sum_{j\neq k} a_{k,j}x_j\right| = \sum_{j\neq k} |a_{k,j}|\,|x_j|$ and also $|x_j| = |x_k|$ for each j. The first equality requires equality in the triangle inequality, meaning that all of the terms $a_{k,j}x_j$ have the same sign, and the second equality requires all of the x_j to have the same absolute value.
   - Since all of the entries of A are positive, these two statements together are equivalent to $x_1 = x_2 = \cdots = x_n$, meaning that $\mathbf{v}$ is a scalar multiple of $(1, 1, \dots, 1)$. This means that the λ-eigenspace is 1-dimensional and spanned by $\mathbf{v} = (1, 1, \dots, 1)$.

   (d) If M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1), show that every eigenvalue λ of M has $|\lambda| \leq 1$. Also show that if M has all entries positive, then the only eigenvalue of M of absolute value 1 is $\lambda = 1$, and the 1-eigenspace has dimension 1. [Hint: Consider M^T.]

   - Note that M^T has rows with nonnegative entries all summing to 1, and it has the same eigenvalues as M. Thus, $\rho_i(M^T) = 1$ for each i, so by (b), we immediately obtain $|\lambda| \leq 1$.
   - Furthermore, since $M^T\mathbf{v} = \mathbf{v}$ where $\mathbf{v} = (1, 1, \dots, 1)$ we see that 1 is indeed an eigenvalue of M^T hence also of M.
   - Therefore, by (c), all other eigenvalues of M have absolute value less than 1, and the 1-eigenspace of M^T (hence also of M) is 1-dimensional.
