E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 8, due Fri Mar 20th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) If $T(\mathbf{v}) = \lambda \mathbf{v}$, then $\mathbf{v}$ is an eigenvector of T.
   (b) Every linear transformation on V has at least one eigenvector.
   (c) If V is finite-dimensional, every linear transformation on V has at least one eigenvector.
   (d) Any two eigenvectors of T are linearly independent.
   (e) The sum of two eigenvectors of T is also an eigenvector of T.
   (f) The sum of two eigenvalues of T is also an eigenvalue of T.
   (g) If two matrices are similar, then they have the same eigenvectors.
   (h) If two matrices have the same eigenvalues, then they are similar.
   (i) If two matrices are similar, then they have the same eigenvalues.
   (j) If $\dim(V) = n$, then T has at most n distinct eigenvalues in F.
   (k) If $\dim(V) = n$, then T has exactly n distinct eigenvalues in F.
   (l) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the 1-eigenspace of A has dimension 2.
   (m) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the only vector $\mathbf{v}$ with $A\mathbf{v} = 3\mathbf{v}$ is $\mathbf{v} = \mathbf{0}$.
   (n) V has a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T if and only if T is diagonalizable.

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2. For each matrix A over each field F, (i) find all eigenvalues of A over F, (ii) find a basis for each eigenspace of A, and (iii) determine whether or not A is diagonalizable over F and if so find an invertible matrix Q and diagonal matrix D such that $D = Q^{-1}AQ$.

   (a) $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{R}$.
   (b) $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Q}$.
   (c) $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{C}$.
   (d) $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$ over $\mathbb{C}$.
   (e) $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$ over $\mathbb{R}$.
   (f) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ over $\mathbb{C}$.

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3. For each operator $T : V \to V$ on each vector space V, (i) find all its eigenvalues and a basis for each eigenspace, and (ii) determine whether the operator is diagonalizable and if so, find a basis for which $[T]_\beta^\beta$ is diagonal:

   (a) The map $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $T(x, y) = (x + 4y, 3x + 5y)$.
   (b) The derivative operator $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $D(p) = p'$.
   (c) The transpose map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(M) = M^T$.

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4. Let F be a field and let L and R be the left shift and right shift operators on infinite sequences of elements of F, defined by $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots)$ and $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

   (a) Find all of the eigenvalues and a basis for each eigenspace of L.
   (b) Find all of the eigenvalues and a basis for each eigenspace of R.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose V is a vector space and $S, T : V \to V$ are linear operators on V.
   (a) If S and T commute (i.e., $ST = TS$), show that S maps each eigenspace of T into itself.
   (b) If $\mathbf{v}$ is an eigenvector of T, show that it is also an eigenvector of T^n for any positive integer n.

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6. Suppose A is an invertible $n \times n$ matrix and that $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is its characteristic polynomial. Note that $a_0 = (-1)^n \det(A)$ is nonzero.
   (a) If $B = -\dfrac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n)$, show that $AB = I_n$. [Hint: Cayley-Hamilton.]
   (b) Show that there exists a polynomial $q(x)$ of degree at most $n-1$ such that $A^{-1} = q(A)$.

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7. Suppose V is finite-dimensional and $T : V \to V$ is a projection, so that $T^2 = T$.
   (a) Show that the only possible eigenvalues of T are 0 and 1.
   (b) Show that T is diagonalizable. [Hint: See homework 5.]
   (c) Suppose A and B are projection maps on V of the same rank. Show that A and B are similar. Deduce that up to similarity, there are $\dim(V) + 1$ different projection maps on V.

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8. The goal of this problem is to explore some counterexamples. Let V be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \dots)$ with only finitely many nonzero terms, with inner product given by $\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^{\infty} a_i b_i$. (Note that this sum converges since only finitely many terms are nonzero.) Let $\mathbf{e}_i$ be the ith unit coordinate vector and observe that $\{\mathbf{e}_i\}_{i\geq 1}$ is an orthonormal basis for V. Now for each $n \geq 2$, let $\mathbf{v}_n = \mathbf{e}_1 - \mathbf{e}_n$ and define $W = \text{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \dots)$.
   (a) Show that $\mathbf{e}_1 \notin W$ so that W is a proper subspace of V, but that $W^\perp = \{\mathbf{0}\}$.
   (b) Show that $W^\perp + W \neq V$ and that $(W^\perp)^\perp \neq W$.
   (c) For any $\mathbf{v} \notin W$, show that there does not exist any choice of $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. Conclude that there is not a well-defined orthogonal projection map of V onto W.
   (d) Show that there exists $\mathbf{w}_n \in W$ such that $||\mathbf{w}_n - \mathbf{e}_1|| = 1/n$ for any positive integer n. Deduce there is no possible best approximation $\mathbf{w} \in W$ to $\mathbf{e}_1$, namely with $||\mathbf{w} - \mathbf{e}_1|| \leq ||\mathbf{w}' - \mathbf{e}_1||$ for all $\mathbf{w}' \in W$. [Hint: Take $\mathbf{w}_n = (1, -1/n, -1/n, \dots, -1/n, 0, 0, \dots)$.]
   (e) Let $T : V \to V$ be the linear transformation defined by setting $T(\mathbf{e}_n) = \sum_{i=1}^{n} \mathbf{e}_i$ for each $i \geq 1$. If T had an adjoint $T^* : V \to V$, show that infinitely many components of $T^*(\mathbf{e}_1)$ would be nonzero. Deduce that T^* cannot exist.

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9. [Challenge] The goal of this problem is to prove various results about eigenvalues of complex matrices and stochastic matrices. Let $A \in M_{n\times n}(\mathbb{C})$, define $\rho_i(A)$ to be the sum of the absolute values of the entries in the ith row of A, and define $\rho(A) = \max_{1\leq i\leq n} \rho_i(A)$.
   (a) Define the ith Gershgorin disk C_i to be the disc in $\mathbb{C}$ centered at $a_{i,i}$ with radius $r_i(A) = \rho_i(A) - |a_{i,i}|$. Prove Gershgorin's disc theorem: every eigenvalue of A is contained in one of the Gershgorin disks of A. [Hint: If $\mathbf{v} = (x_1, \dots, x_n)$ is an eigenvector where x_k has the largest absolute value among the entries of $\mathbf{v}$, show that $|\lambda x_k - a_{k,k}x_k| \leq r_i(A)\,|x_k|$ by noting that λx_k is the kth component of $A\mathbf{v}$.]
   (b) For any eigenvalue λ of $A \in M_{n\times n}(\mathbb{C})$, prove that $|\lambda| \leq \rho(A)$.
   (c) Prove that if $A \in M_{n\times n}(\mathbb{R})$ has positive entries and there exists an eigenvalue λ such that $|\lambda| = \rho(A)$, then $\lambda = \rho(A)$ and the λ-eigenspace is 1-dimensional and spanned by the vector $\mathbf{v} = (1, 1, \dots, 1)$. [Hint: Analyze when equality can hold in (a) and (b).]
   (d) If M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1), show that every eigenvalue λ of M has $|\lambda| \leq 1$. Also show that if M has all entries positive, then the only eigenvalue of M of absolute value 1 is $\lambda = 1$, and the 1-eigenspace has dimension 1. [Hint: Consider M^T.]
