

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V with scalar field F with $\mathbf{v}, \mathbf{w} \in V$, and let W be a subspace of V . Identify each of the following statements as true or false:
 - (a) If V is finite-dimensional and W is any subspace of V , then $\dim(W) = \dim(W^\perp)$.
 - False: the correct formula is $\dim(W) + \dim(W^\perp) = \dim(V)$.
 - (b) If \mathbf{w}^\perp is a vector in W^\perp , then the orthogonal projection of \mathbf{w}^\perp onto W is \mathbf{w}^\perp itself.
 - False: the orthogonal projection of a vector \mathbf{w}^\perp in W^\perp onto W is zero.
 - (c) If $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is an orthonormal basis of W , then $\mathbf{w} = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \dots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$ is the orthogonal projection of \mathbf{v} into W .
 - True: this is the orthogonal projection formula we proved.
 - (d) If V is finite-dimensional, $\mathbf{v} \in V$, and W is any subspace of V , the vector $\mathbf{w} \in W$ minimizing $\|\mathbf{v} - \mathbf{w}\|$ is the orthogonal projection of \mathbf{v} into W .
 - True: this is the best-approximation property of the orthogonal projection.
 - (e) If $T : V \rightarrow V$ is linear, then the adjoint of T exists and is unique.
 - False: the adjoint does not always necessarily exist over an arbitrary vector space. (If it does exist, then it is unique.)
 - (f) If $T : V \rightarrow V$ is linear and V is finite-dimensional, then the adjoint of T exists and is unique.
 - True: we proved that the adjoint always exists over finite-dimensional vector spaces.
 - (g) If $T : V \rightarrow F$ is linear and V is finite-dimensional, then there exists $\mathbf{w} \in V$ such that $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v} \in V$.
 - True: this is the version of the Riesz representation theorem we established.
 - (h) For any $S, T : V \rightarrow V$ such that S^* and T^* exist, we have $(S + iT)^* = S^* + iT^*$.
 - False: the correct formula is $(S + iT)^* = S^* - iT^*$.
 - (i) For any $S, T : V \rightarrow V$ such that S^* and T^* exist, we have $(ST)^* = S^*T^*$.
 - False: the correct formula is $(ST)^* = T^*S^*$.
 - (j) If $A\mathbf{x} = \mathbf{c}$ is an inconsistent linear system, the best approximation of a solution is an $\hat{\mathbf{x}}$ satisfying $A^*\hat{\mathbf{x}} = A^*\mathbf{c}$.
 - False: the correct equation to solve is the normal equation $(A^*A)\hat{\mathbf{x}} = A^*\mathbf{c}$.
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2. Calculate the following things:

- (a) A basis for W^\perp , if $W = \text{span}[(1, 1, 1, 1), (2, 3, 4, 1)]$ inside \mathbb{R}^4 under the dot product.
 - The orthogonal complement corresponds to the nullspace of the matrix whose rows are the given vectors.
 - Row-reducing $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ yields the reduced row-echelon form $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$.
 - From the reduced row-echelon form, we see that $\{(-2, 1, 0, 1), (1, -2, 1, 0)\}$ is a basis for the nullspace and hence of W^\perp .
- (b) The orthogonal decomposition $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ of $\mathbf{v} = (2, 0, 11)$ into $W = \text{span}[\frac{1}{3}(1, 2, 2), \frac{1}{3}(2, -2, 1)]$ inside \mathbb{R}^3 under the dot product. Also, verify the relation $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2 + \|\mathbf{w}^\perp\|^2$.
 - Notice that the vectors $\mathbf{e}_1 = \frac{1}{3}(1, 2, 2)$ and $\mathbf{e}_2 = \frac{1}{3}(2, -2, 1)$ form an orthonormal basis for W .
 - Thus, the orthogonal projection is $\mathbf{w} = \text{proj}_W(\mathbf{v}) = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{v}, \mathbf{e}_2 \rangle \mathbf{e}_2 = 8\mathbf{e}_1 + 5\mathbf{e}_2 = \boxed{(6, 2, 7)}$.

- We see that $\mathbf{w}^\perp = \mathbf{v} - \mathbf{w} = (-4, -2, 4)$ is orthogonal to both \mathbf{e}_1 and \mathbf{e}_2 so it is indeed in W^\perp .
 - Furthermore, $\|\mathbf{v}\|^2 = 125$, while $\|\mathbf{w}\|^2 = 89$ and $\|\mathbf{w}^\perp\|^2 = 36$, so indeed $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2 + \|\mathbf{w}^\perp\|^2$.
- (c) The quadratic polynomial $p(x) \in P_2(\mathbb{R})$ that minimizes the expression $\int_0^1 [p(x) - \sqrt{x}]^2 dx$.
- The desired polynomial is the orthogonal projection of $f(x) = \sqrt{x}$ into $W = P_2(\mathbb{R})$. Using Gram-Schmidt we can find an orthogonal basis for W , which yields $\mathbf{e}_1 = 1$, $\mathbf{e}_2 = -1 + 2x$, $\mathbf{e}_3 = 1 - 6x + 6x^2$.
 - Using the orthogonal basis, we get the projection

$$\mathbf{w} = \frac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \mathbf{e}_1 + \frac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle} \mathbf{e}_2 + \frac{\langle \mathbf{v}, \mathbf{e}_3 \rangle}{\langle \mathbf{e}_3, \mathbf{e}_3 \rangle} \mathbf{e}_3 = \frac{2}{3} \mathbf{e}_1 + \frac{2}{5} \mathbf{e}_2 - \frac{2}{21} \mathbf{e}_3 = \boxed{\frac{6}{35} + \frac{48}{35}x - \frac{4}{7}x^2}.$$
- (d) The least-squares solution to the inconsistent system $x + 3y = 9$, $3x + y = 5$, $x + y = 2$.
- We have $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$. Since A clearly has rank 2, $A^T A$ will be invertible and there will be a unique least-squares solution.
 - We compute $A^* A = \begin{bmatrix} 11 & 7 \\ 7 & 11 \end{bmatrix}$, which is indeed invertible and has inverse $(A^* A)^{-1} = \frac{1}{72} \begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix}$.
 - The least-squares solution is therefore $\hat{\mathbf{x}} = (A^* A)^{-1} A^* \mathbf{c} = \boxed{\begin{bmatrix} 2/3 \\ 8/3 \end{bmatrix}}$.
- (e) The least-squares line $y = a + bx$ approximating the points $\{(4, 7), (11, 21), (15, 29), (19, 35), (30, 49)\}$. (Give three decimal places.)
- We seek the least-squares solution for $A\mathbf{x} = \mathbf{c}$, where $A = \begin{bmatrix} 1 & 4 \\ 1 & 11 \\ 1 & 15 \\ 1 & 19 \\ 1 & 30 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 7 \\ 21 \\ 29 \\ 35 \\ 49 \end{bmatrix}$.
 - We compute $A^* A = \begin{bmatrix} 5 & 79 \\ 79 & 1623 \end{bmatrix}$, so the least-squares solution is $\hat{\mathbf{x}} = (A^* A)^{-1} A^* \mathbf{c} \approx \begin{bmatrix} 2.856 \\ 1.604 \end{bmatrix}$.
 - Thus, to three decimal places, the desired line is $y = \boxed{1.604x + 2.856}$.
- (f) The least-squares quadratic $y = a + bx + cx^2$ approximating the points $\{(-2, 22), (-1, 11), (0, 4), (1, 3), (2, 13)\}$. (Give three decimal places.)
- We seek the least-squares solution for $A\mathbf{x} = \mathbf{c}$, with $A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 22 \\ 11 \\ 4 \\ 3 \\ 13 \end{bmatrix}$.
 - We compute $A^* A = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$, so the least-squares solution is $\hat{\mathbf{x}} = (A^* A)^{-1} A^* \mathbf{c} \approx \begin{bmatrix} 3.743 \\ -2.6 \\ 3.429 \end{bmatrix}$.
 - Thus, the desired quadratic polynomial is $y = \boxed{3.743 - 2.6x + 3.429x^2}$.

3. Let V be an inner product space with scalar field F . The goal of this problem is to prove the so-called “polarization identities”.

- (a) If $F = \mathbb{R}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \|\mathbf{v} + \mathbf{w}\|^2 - \frac{1}{4} \|\mathbf{v} - \mathbf{w}\|^2$.
- We just expand the norms on the right-hand side: $\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle - \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle = [\langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] - [\langle \mathbf{v}, \mathbf{v} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] = 4\langle \mathbf{v}, \mathbf{w} \rangle$ so dividing by 4 yields the claimed result.
- (b) If $F = \mathbb{C}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \|\mathbf{v} + \mathbf{w}\|^2 + \frac{i}{4} \|\mathbf{v} + i\mathbf{w}\|^2 - \frac{1}{4} \|\mathbf{v} - \mathbf{w}\|^2 - \frac{i}{4} \|\mathbf{v} - i\mathbf{w}\|^2$.

- As in part (a) we just expand the norms on the right-hand side:

$$\begin{aligned}
\sum_{k=1}^4 i^k \|\mathbf{v} + i^k \mathbf{w}\|^2 &= \|\mathbf{v} + \mathbf{w}\|^2 + i\|\mathbf{v} + i\mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 - i\|\mathbf{v} - i\mathbf{w}\|^2 \\
&= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle + i\langle \mathbf{v} + i\mathbf{w}, \mathbf{v} + i\mathbf{w} \rangle - \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle - i\langle \mathbf{v} - i\mathbf{w}, \mathbf{v} - i\mathbf{w} \rangle \\
&= [\langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] + i[\langle \mathbf{v}, \mathbf{v} \rangle - i\langle \mathbf{v}, \mathbf{w} \rangle + i\langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] \\
&\quad - [\langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] - i[\langle \mathbf{v}, \mathbf{v} \rangle + i\langle \mathbf{v}, \mathbf{w} \rangle - i\langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] \\
&= 4\langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

and so dividing by 4 yields the claimed result.

4. Let V be a finite-dimensional inner product space and W be a subspace of V .

(a) Prove that $W \cap W^\perp = \{\mathbf{0}\}$ and deduce that $V = W \oplus W^\perp$. [Hint: Use $\dim(W) + \dim(W^\perp) = \dim(V)$.]

- Suppose $\mathbf{w} \in W \cap W^\perp$. Then $\langle \mathbf{w}, \mathbf{w} \rangle = 0$ since the inner product of any vector in W with any vector in W^\perp is 0. But then property [I3] of the inner product immediately implies $\mathbf{w} = \mathbf{0}$, so $W \cap W^\perp = \{\mathbf{0}\}$.
- For the second statement, per the hint, since $\dim(W) + \dim(W^\perp) = \dim(V)$ and $W \cap W^\perp = \{\mathbf{0}\}$ we must have $W + W^\perp = V$ (in fact we already proved this by showing that the union of a basis of W and a basis of W^\perp gives a basis for V). Hence by the definition of direct sum, we have $V = W \oplus W^\perp$.

(b) Let $T : V \rightarrow W$ be the function defined by setting $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ for $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$. Prove that T is linear, that $T^2 = T$, that $\text{im}(T) = W$, and that $\text{ker}(T) = W^\perp$. Conclude that T is projection onto the subspace W with kernel W^\perp .

- Suppose $\mathbf{v}_1 = \mathbf{w}_1 + \mathbf{w}_1^\perp$ and $\mathbf{v}_2 = \mathbf{w}_2 + \mathbf{w}_2^\perp$ where the $\mathbf{w}_i \in W$ and the $\mathbf{w}_i^\perp \in W^\perp$.
- Then for any scalar c , we see $\mathbf{v}_1 + c\mathbf{v}_2 = (\mathbf{w}_1 + c\mathbf{w}_2) + (\mathbf{w}_1^\perp + c\mathbf{w}_2^\perp)$ where $\mathbf{w}_1 + c\mathbf{w}_2 \in W$ and $\mathbf{w}_1^\perp + c\mathbf{w}_2^\perp \in W^\perp$ since these are both subspaces. Thus, by uniqueness of orthogonal decomposition, this is the orthogonal decomposition of $\mathbf{v}_1 + c\mathbf{v}_2$.
- Then $T(\mathbf{v}_1 + c\mathbf{v}_2) = \mathbf{w}_1 + c\mathbf{w}_2 = T(\mathbf{v}_1) + cT(\mathbf{v}_2)$ so T is linear.
- Also, $T^2(\mathbf{v}) = T(T(\mathbf{v})) = T(\mathbf{w}) = \mathbf{w}$ since we can clearly write $\mathbf{w} = \mathbf{w} + \mathbf{0}$ so $T(\mathbf{w}) = \mathbf{w}$ again by uniqueness. Hence $T^2(\mathbf{v}) = T(\mathbf{v})$ for every $\mathbf{v} \in V$, so T is a projection map.

5. Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \rightarrow V$ is a linear transformation possessing an adjoint T^* . We say T is Hermitian (or self-adjoint) if $T = T^*$, and that T is skew-Hermitian if $T = -T^*$.

(a) Show that T is Hermitian if and only if iT is skew-Hermitian.

- Note $(iT)^* = -iT^*$ so $T = T^*$ if and only if $(iT)^* = -iT$.

(b) Show that $T + T^*$, T^*T , and TT^* are all Hermitian, while $T - T^*$ is skew-Hermitian.

- Note $(T + T^*)^* = T^* + T^{**} = T + T^*$, $(T^*T)^* = T^*T^{**} = T^*T$, and $(TT^*)^* = T^{**}T^* = TT^*$.
- Also, $(T - T^*)^* = T^* - T^{**} = T^* - T$.

(c) Show that T can be written as $T = S_1 + iS_2$ for unique Hermitian transformations S_1 and S_2 .

- In such a case we would necessarily have $T^* = (S_1 + iS_2)^* = S_1^* - iS_2^* = S_1 - iS_2$.
- Solving for S_1 and S_2 in terms of T and T^* then yields $S_1 = \frac{1}{2}(T + T^*)$ and $S_2 = \frac{1}{2i}(T - T^*)$, so these are the only possible choices.
- On the other hand, by (a) and (b), we see that these S_1 and S_2 are in fact Hermitian, so these are the unique choices.

(d) Suppose T is Hermitian. Prove that $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is a real number for any vector \mathbf{v} .

- If $T^* = T$ then $\langle T\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, T^*\mathbf{v} \rangle = \langle \mathbf{v}, T\mathbf{v} \rangle = \overline{\langle T\mathbf{v}, \mathbf{v} \rangle}$, so $\langle T\mathbf{v}, \mathbf{v} \rangle$ equals its conjugate hence is real.

6. Suppose V is an inner product space over the field F (where $F = \mathbb{R}$ or \mathbb{C}) and $T : V \rightarrow V$ is linear. We say T is a “distance-preserving map” on V if $\|T\mathbf{v}\| = \|\mathbf{v}\|$ for all \mathbf{v} in V , and we say T is a “pairing-preserving map” on V if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{w} \rangle$ for all \mathbf{v} and \mathbf{w} in V .

(a) Prove that T is distance-preserving if and only if it is pairing-preserving. [Hint: Use problem 3.]

- Clearly if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{w} \rangle$ then $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \langle T\mathbf{v}, T\mathbf{v} \rangle = \|T\mathbf{v}\|^2$.
- The converse follows by using the polarization identities from problem 3 to recover the inner product from the norm.
- For example, if $F = \mathbb{R}$ we have $\langle T\mathbf{v}, T\mathbf{w} \rangle = \frac{1}{4} \|T(\mathbf{v} + \mathbf{w})\|^2 - \frac{1}{4} \|T(\mathbf{v} - \mathbf{w})\|^2 = \frac{1}{4} \|\mathbf{v} + \mathbf{w}\|^2 - \frac{1}{4} \|\mathbf{v} - \mathbf{w}\|^2 = \langle \mathbf{v}, \mathbf{w} \rangle$, and similarly in the complex case.

A map $T : V \rightarrow V$ satisfying the distance- and pairing-preserving conditions is called a (linear) isometry.

(b) Show that the transformations $S, T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $S(x, y, z) = (z, -x, y)$ and $T(x, y, z) = \frac{1}{3}(x + 2y + 2z, 2x + y - 2z, 2x - 2y + z)$ are both isometries under the usual dot product.

- We simply compute $\|S(x, y, z)\| = \sqrt{z^2 + x^2 + y^2} = \|(x, y, z)\|$, and $\|T(x, y, z)\| = \frac{1}{9} [(x + 2y + 2z)^2 + (2x + y - 2z)^2 + (2x - 2y + z)^2] = \sqrt{x^2 + y^2 + z^2} = \|(x, y, z)\|$.
- Thus, S and T both preserve norms, so by the above, they are isometries.

(c) Show that isometries are one-to-one.

- Suppose T is an isometry. If $\mathbf{v} \in \ker(T)$ then $\|\mathbf{v}\| = \|T(\mathbf{v})\| = 0$, so $\mathbf{v} = 0$ by [I3]. Thus T is one-to-one.

(d) Show that isometries preserve orthogonal and orthonormal sets.

- Suppose T is an isometry. If $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ then $\langle T\mathbf{v}, T\mathbf{w} \rangle = 0$.
- Thus, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ is an orthogonal set, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots\}$ is also orthogonal.
- Furthermore, since $\|\mathbf{v}\| = \|T(\mathbf{v})\|$, T also preserves orthonormal sets.

(e) Suppose T^* exists. Prove that T is an isometry if and only if T^*T is the identity transformation.

- Observe that $\langle \mathbf{v}, \mathbf{w} \rangle - \langle T\mathbf{v}, T\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}, T^*T\mathbf{w} \rangle = \langle \mathbf{v}, (I - T^*T)\mathbf{w} \rangle$.
- Thus, setting $\mathbf{v} = (I - T^*T)\mathbf{w}$ shows that the left-hand side is identically zero if and only if $(I - T^*T)\mathbf{w}$ is identically zero, if and only if $T^*T = I$.

(f) We say that a matrix $A \in M_{n \times n}(F)$ is unitary if $A^{-1} = A^*$. Show that the isometries of F^n (with its usual inner product) are precisely those maps given by left-multiplication by a unitary matrix.

- This is simply the matrix version of (e): the isometries are the linear transformations with $A^*A = I_n$, and $A^*A = I_n$ is equivalent to saying that $A^{-1} = A^*$.

Remark: Notice that $A \in M_{n \times n}(\mathbb{C})$ is unitary if and only if the columns of A are an orthonormal basis of \mathbb{C}^n . Thus, the result of part (f) can equivalently be thought of as saying that the distance-preserving maps on \mathbb{C}^n (or \mathbb{R}^n) are simply changes of basis from one orthonormal basis (the columns of A) to another (the standard basis).

7. Let $V = C[0, 2\pi]$ with inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x) dx$. Also define $\varphi_0(x) = \frac{1}{\sqrt{2\pi}}$, and for positive integers k set $\varphi_{2k-1}(x) = \frac{1}{\sqrt{\pi}} \cos(kx)$ and $\varphi_{2k}(x) = \frac{1}{\sqrt{\pi}} \sin(kx)$.

(a) Show that $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ is an orthonormal set in V . [Hint: Use the product-to-sum identities.]

- First observe that $\int_0^{2\pi} \sin(nx) dx = 0 = \int_0^{2\pi} \cos(nx) dx$ for any integer $n \neq 0$. Thus, $\langle \varphi_0, \varphi_k \rangle = 0$ for any $k > 0$.
- Furthermore, using the product-to-sum identities, we can write

$$\begin{aligned}\varphi_{2a}\varphi_{2b} &= \frac{1}{\pi} \sin(ax) \sin(bx) = \frac{1}{2\pi} [\cos(a-b)x - \cos(a+b)x] \\ \varphi_{2a-1}\varphi_{2b} &= \frac{1}{\pi} \cos(ax) \sin(bx) = \frac{1}{2\pi} [\sin(a-b)x - \sin(a+b)x] \\ \varphi_{2a-1}\varphi_{2b-1} &= \frac{1}{\pi} \cos(ax) \cos(bx) = \frac{1}{2\pi} [\cos(a-b)x + \cos(a+b)x]\end{aligned}$$

and so when $a \neq b$, each inner product $\langle \varphi_{2a}, \varphi_{2b} \rangle$, $\langle \varphi_{2a-1}, \varphi_{2b} \rangle$, and $\langle \varphi_{2a-1}, \varphi_{2b-1} \rangle$ is zero because both terms integrate to zero (the second also integrates to zero when $a = b$).

- Furthermore, we have $\langle \varphi_0, \varphi_0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} 1 dx = 1$, $\langle \varphi_{2k-1}, \varphi_{2k-1} \rangle = \frac{1}{\pi} \int_0^{2\pi} \cos^2(kx) dx = 1$, and $\langle \varphi_{2k}, \varphi_{2k} \rangle = \frac{1}{\pi} \int_0^{2\pi} \sin^2(kx) dx = 1$. Thus, the set is orthonormal.

(b) Let $f(x) = x$. Find $\|f\|$ and $\langle f, \varphi_n \rangle$ for each $n \geq 0$. (You don't need to give details of the integral evaluations, just the resulting values.)

- We compute $\|f\| = \sqrt{\int_0^{2\pi} x^2 dx} = \sqrt{\frac{8\pi^3}{3}}$, $\langle \varphi_0, f \rangle = \int_0^{2\pi} \frac{x}{\sqrt{2\pi}} dx = \sqrt{2\pi^3}$,
with $\langle \varphi_{2k-1}, f \rangle = \int_0^{2\pi} \frac{x}{\sqrt{\pi}} \cos(kx) dx = 0$, and $\langle \varphi_{2k}, f \rangle = \int_0^{2\pi} \frac{x}{\sqrt{\pi}} \sin(kx) dx = -\frac{2\sqrt{\pi}}{k}$ for $k \geq 1$.

(c) With $f(x) = x$, assuming that $f(x) = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k(x)$, derive Leibniz's formula $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. [Hint: Set $x = \pi/2$.]

- By part (b), the formula yields $f(\pi/2) = \pi - \sum_{k=1}^{\infty} \frac{2}{k} \sin(k\pi/2) = \pi - \left[2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \dots\right]$. Since $f(\pi/2) = \pi/2$, rearranging yields $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ as claimed.

(d) With $f(x) = x$, assuming that $\|f\|^2 = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle^2$, find the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

- By part (b), the formula yields $2\pi^3 + \sum_{k=1}^{\infty} \left(-\frac{2\sqrt{\pi}}{k}\right)^2 = \frac{8\pi^3}{3}$, so that $\sum_{k=1}^{\infty} \frac{4\pi}{k^2} = \frac{2\pi^3}{3}$.
- Dividing by 4π yields $\sum_{k=1}^{\infty} \frac{1}{k^2} = \boxed{\frac{\pi^2}{6}}$. (This is, in fact, the actual value of this sum!)

Remarks: The identity $\|f\|^2 = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle^2$ is known as Parseval's identity. The problem of computing the value of the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is known as the Basel problem. The correct value was (famously) first identified by Euler, who evaluated the sum by decomposing the function $\frac{\sin \pi x}{\pi x}$ as the infinite product $\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$ and then comparing the power series coefficients of both sides.

8. [Challenge] The goal of this problem is to give an example of an inner product space that has no orthonormal basis. Let $V = \ell^2(\mathbb{R})$ be the vector space of infinite real sequences $\{a_i\}_{i \geq 1} = (a_1, a_2, \dots)$ such that $\sum_{i=1}^{\infty} a_i^2$ is finite, under componentwise addition and scalar multiplication.

(a) Show that the pairing $\langle \{a_i\}_{i \geq 1}, \{b_i\}_{i \geq 1} \rangle = \sum_{i=1}^{\infty} a_i b_i$ is an inner product on V . (Make sure to justify why this sum converges.)

- First, we must justify why the pairing is well defined: starting with the Cauchy-Schwarz inequality $(\sum_{i=1}^n a_i b_i)^2 \leq \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2$, if we take the limit as $n \rightarrow \infty$, then we obtain $(\sum_{i=1}^{\infty} a_i b_i)^2 \leq \sum_{i=1}^{\infty} a_i^2 \cdot \sum_{i=1}^{\infty} b_i^2$. But the right-hand side is finite by the assumption that $\{a_i\}_{i \geq 1}$ and $\{b_i\}_{i \geq 1}$ are elements of V , and so this means $|\sum_{i=1}^{\infty} a_i b_i|$ is finite (i.e., the sum converges).
- Now we just verify the three parts of the definition of an inner product. Let $\mathbf{v} = \{a_i\}_{i \geq 1}$, $\mathbf{v}' = \{a'_i\}_{i \geq 1}$, $\mathbf{w} = \{b_i\}_{i \geq 1}$.
- [I1]: We have $\langle \mathbf{v} + \alpha \mathbf{v}', \mathbf{w} \rangle = \sum_{i=1}^{\infty} (a_i + \alpha a'_i) b_i = \sum_{i=1}^{\infty} a_i b_i + \alpha \sum_{i=1}^{\infty} a'_i b_i = \langle \mathbf{v}, \mathbf{w} \rangle + \alpha \langle \mathbf{v}', \mathbf{w} \rangle$.
- [I2]: We have $\langle \mathbf{w}, \mathbf{v} \rangle = \sum_{i=1}^{\infty} b_i a_i = \sum_{i=1}^{\infty} a_i b_i = \langle \mathbf{v}, \mathbf{w} \rangle$.
- [I3]: We have $\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{\infty} a_i^2$ which is a sum of squares hence always nonnegative, and it is zero only when all $a_i = 0$, which is to say, when $\mathbf{v} = \mathbf{0}$.

(b) Let $\mathbf{v}_i \in V$ be the sequence with a 1 in the i th component and 0s elsewhere. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \dots\}$ is an orthonormal set in V and that the only vector \mathbf{w} orthogonal to all of the \mathbf{v}_i is the zero vector. Deduce that S is a maximal orthonormal set of V that is not a basis of V .

- It is obvious that S is orthonormal, since $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$ and $\|\mathbf{v}_i\| = 1$ for all i .
- If $\mathbf{w} = \{b_i\}_{i \geq 1}$, then $\langle \mathbf{w}, \mathbf{v}_i \rangle = b_i$ since the only term that survives in the sum is $b_i \cdot 1$ in the i th term. So if $\langle \mathbf{w}, \mathbf{v}_i \rangle = 0$ for all i , then each $b_i = 0$ so that $\mathbf{w} = \mathbf{0}$.
- This means that S is a maximal orthonormal subset of V , since it is orthonormal and there are no vectors that could be added to it to preserve orthonormality (since the only vector orthogonal to all of S is the zero vector, by the above).
- On the other hand, S is not a basis of V , since for example the vector $\{1/2^i\}_{i \geq 1}$, which is in V since $\sum_{i=1}^{\infty} (1/2^i)^2 = 1/3$ is finite, cannot be written as a finite linear combination of the vectors in S .

Part (b) shows that Gram-Schmidt does not necessarily construct an orthonormal basis of V . In fact, V has no orthonormal basis at all.

(c) Suppose V has an orthonormal basis $\{\mathbf{e}_i\}_{i \in I}$ for some indexing set I (which is necessarily infinite), and choose a countably infinite subset $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \dots$. Show that the sum $\mathbf{v} = \sum_{k=1}^{\infty} 2^{-k} \mathbf{e}_k$ is a well-defined vector in V that cannot be written as a (finite) linear combination of the basis $\{\mathbf{e}_i\}_{i \in I}$. [Hint: Show that $\|\mathbf{v}\|^2 = \lim_{n \rightarrow \infty} \|\sum_{k=1}^n 2^{-k} \mathbf{e}_k\|^2$ is finite.]

- We need to show that the sum for \mathbf{v} converges to a vector in V . Clearly, since the vectors \mathbf{e}_i in the sum all have length 1, each of their coordinates is at most 1, so the i th coordinate of $\sum_{k=1}^{\infty} 2^{-k} \mathbf{e}_k$ is at most $\sum_{k=1}^{\infty} 2^{-k} = 1$. So the coordinates of the sum certainly all converge to a finite value.
- Furthermore, the sum is in fact in V : since the vectors \mathbf{e}_i are orthonormal, we have $\|\mathbf{v}\|^2 = \lim_{n \rightarrow \infty} \|\sum_{k=1}^n 2^{-k} \mathbf{e}_k\|^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n 2^{-2k} = \lim_{n \rightarrow \infty} \frac{1}{3}(1 - 1/2^{-2n}) = \frac{1}{3}$, so in fact $\|\mathbf{v}\|^2$ is finite and so \mathbf{v} is in V .
- However, we claim that \mathbf{v} cannot be written as a finite linear combination of the basis elements \mathbf{e}_i .
- To see this, first note that since $\{\mathbf{e}_i\}_{i \in I}$ is an orthonormal basis of V , the inner product $\langle \mathbf{v}, \mathbf{e}_i \rangle$ gives the coefficient of \mathbf{e}_i in the decomposition of \mathbf{v} . But since $\mathbf{v} = \sum_{k=1}^{\infty} 2^{-k} \mathbf{e}_k$, we see that $\langle \mathbf{v}, \mathbf{e}_k \rangle = 2^{-k}$ is nonzero for infinitely many of the basis vectors \mathbf{e}_k . This precludes the possibility that \mathbf{v} is a finite linear combination of the basis elements \mathbf{e}_i , which is a contradiction since $\{\mathbf{e}_i\}_{i \in I}$ was assumed to be a basis.

Remark: The point here is that because our definition of span and basis only allows us to use finite linear combinations, these definitions are not well suited to handle infinite-dimensional spaces like $\ell^2(\mathbb{R})$. However, it is possible (by exploiting the fact that ℓ^2 is a topologically-complete metric space) to deal with these issues and define a ‘‘Schauder basis’’ that allows the use of infinite sums, which amounts to viewing ℓ^2 as a Hilbert space.