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1. Identify each of the following statements as true or false:

   (a) If V has ordered bases β, γ and $T : V \to V$ has an inverse T^{-1}, then $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$.

   - $\boxed{\text{False}}$: the correct formula is $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$, since $[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = [I]_\beta^\beta = I_n$.

   (b) If V has ordered bases β, γ then for any $T : V \to V$ there exists an invertible Q with $[T]_\gamma^\gamma = Q[T]_\beta^\beta Q^{-1}$.

   - $\boxed{\text{True}}$: if we take Q to be the change-of-basis matrix $[I]_\gamma^\beta$, then $Q^{-1} = [I]_\beta^\gamma$ so $Q^{-1}[T]_\beta^\beta Q = [I]_\beta^\gamma [T]_\beta^\beta [I]_\gamma^\beta = [T]_\gamma^\gamma$.

   (c) An inner product is linear in each of its components.

   - $\boxed{\text{False}}$: a complex inner product is not linear in its second component.

   (d) There is exactly one inner product on $\mathbb{R}^n$.

   - $\boxed{\text{False}}$: there are numerous inner products on $\mathbb{R}^n$.

   (e) In any inner product space, $\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$.

   - $\boxed{\text{False}}$: the correct statement is $\overline{\langle \mathbf{w}, \mathbf{v} \rangle} = \langle \mathbf{v}, \mathbf{w} \rangle$.

   (f) In any inner product space, $||\mathbf{v} + \mathbf{w}|| \geq ||\mathbf{v}|| + ||\mathbf{w}||$.

   - $\boxed{\text{False}}$: the correct statement is $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.

   (g) In any inner product space, if $\langle \mathbf{v}, 2\mathbf{v} \rangle = 0$ then $\mathbf{v} = \mathbf{0}$.

   - $\boxed{\text{True}}$: $\langle \mathbf{v}, 2\mathbf{v} \rangle = 2 \langle \mathbf{v}, \mathbf{v} \rangle$ so $\langle \mathbf{v}, 2\mathbf{v} \rangle = 0$ implies $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ which only occurs for $\mathbf{v} = \mathbf{0}$.

   (h) In any inner product space, if $\langle \mathbf{v}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{y} \rangle$ then $\mathbf{x} = \mathbf{y}$.

   - $\boxed{\text{False}}$: for example $(1, 1) \cdot (2, 3) = 5 = (1, 1) \cdot (4, 1)$ but $(2, 3) \neq (4, 1)$.

   (i) In any inner product space, for a fixed $\mathbf{w} \in V$, the map $T : V \to F$ with $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ is linear.

   - $\boxed{\text{True}}$: we have $T(\mathbf{v}_1 + \alpha\mathbf{v}_2) = \langle \mathbf{v}_1 + \alpha\mathbf{v}_2, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle + \alpha \langle \mathbf{v}_2, \mathbf{w} \rangle = T(\mathbf{v}_1) + \alpha T(\mathbf{v}_2)$.

   (j) In any inner product space, for a fixed $\mathbf{w} \in V$, the map $T : V \to F$ with $T(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle$ is linear.

   - $\boxed{\text{False}}$: we have $T(\alpha\mathbf{v}) = \langle \mathbf{w}, \alpha\mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{w}, \mathbf{v} \rangle = \overline{\alpha} T(\mathbf{v})$ rather than $\alpha T(\mathbf{v})$.

   (k) The Cauchy-Schwarz inequality holds in every inner product space.

   - $\boxed{\text{True}}$: we proved the Cauchy-Schwarz inequality in an arbitrary inner product space.

   (l) The triangle inequality holds in real inner product spaces but not complex inner product spaces.

   - $\boxed{\text{False}}$: the triangle inequality holds in every inner product space.

   (m) An orthogonal set of vectors is linearly independent.

   - $\boxed{\text{False}}$: the set $\{(0, 0, 0), (1, 0, 0)\}$ is orthogonal but not linearly independent.

   (n) An orthonormal set of vectors is linearly independent.

   - $\boxed{\text{True}}$: we showed any set of nonzero orthogonal vectors is linearly independent, and an orthonormal set is orthogonal and cannot include the zero vector (since its norm is 0).

   (o) Every finite-dimensional inner product space has an orthonormal basis.

   - $\boxed{\text{True}}$: we can construct an orthonormal basis via Gram-Schmidt.

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2. For each of the following pairings, determine (with brief justification) whether or not it is an inner product on the given vector space:

   (a) The pairing $\langle A, B\rangle = \mathrm{tr}(A+B)$ on $M_{2\times 2}(\mathbb{R})$.

   - This is $\boxed{\text{not an inner product}}$, as it fails both [I1] and [I3]: e.g., $\langle A, A\rangle = \mathrm{tr}(2A)$ can be negative.

   (b) The pairing $\langle (a,b),(c,d)\rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{R}^2$.

   - This $\boxed{\text{is an inner product}}$: it is easy to see it satisfies [I1] and [I2], and for [I3] notice that $\langle (a,b),(a,b)\rangle = 5a^2 + 6ab + 4b^2 = 5(a + \frac{3}{5}b)^2 + \frac{11}{5}b^2$, which is nonnegative since it is a sum of squares and is only zero when $a + \frac{3}{5}b = b = 0$.

   (c) The pairing $\langle (a,b),(c,d)\rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{C}^2$.

   - This is $\boxed{\text{not an inner product}}$, as it fails [I2] and [I3] because it is not conjugate-symmetric.

   (d) The pairing $\langle (a,b),(c,d)\rangle = ac$ on $\mathbb{R}^2$.

   - This is $\boxed{\text{not an inner product}}$, since it fails the last part of [I3]: we have $\langle (0,1),(0,1)\rangle = 0$, but $(0,1)$ is not the zero vector. (The first two conditions do hold, however.)

   (e) The pairing $\langle f, g\rangle = \int_0^1 f'(x)g(x)\,dx$ on $C[0,1]$.

   - This is $\boxed{\text{not an inner product}}$, as it fails [I2] and [I3]. For example, $\left\langle x, x^2\right\rangle = \dfrac{1}{3}$ while $\left\langle x^2, x\right\rangle = \dfrac{2}{3}$. (The condition [I1] does hold, however.)

   (f) The pairing $\langle f, g\rangle = f(0)g(0) + f(1)g(1) + f(2)g(2)$ on $P_2(\mathbb{R})$.

   - This is $\boxed{\text{is an inner product}}$: it is easy to see it satisfies [I1] and [I2], and for [I3] notice $\langle f, f\rangle = f(0)^2 + f(1)^2 + f(2)^2$ which is nonnegative and is only zero when $f(0) = f(1) = f(2)$, but the only quadratic polynomial with this property is the zero polynomial.

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3. For each pair of vectors $\mathbf{v}, \mathbf{w}$ in the given inner product space, compute $\langle \mathbf{v}, \mathbf{w}\rangle$, $||\mathbf{v}||$, $||\mathbf{w}||$, and $||\mathbf{v}+\mathbf{w}||$, and verify the Cauchy-Schwarz and triangle inequalities for $\mathbf{v}$ and $\mathbf{w}$:

   (a) $\mathbf{v} = (1,2,2,4)$ and $\mathbf{w} = (4,1,4,4)$ in $\mathbb{R}^4$ with the standard inner product.

   - $\langle \mathbf{v}, \mathbf{w}\rangle = 1\cdot 4 + 2\cdot 1 + 2\cdot 4 + 4\cdot 4 = 30$, $||\mathbf{v}|| = \sqrt{1^2+2^2+2^2+4^2} = 5$, $||\mathbf{w}|| = \sqrt{4^2+1^2+4^2+4^2} = 7$, and $||\mathbf{v}+\mathbf{w}|| = \sqrt{5^2+3^2+6^2+8^2} = \sqrt{134}$.
   - Cauchy-Schwarz holds since $|30| \le 5\cdot 7$, and the triangle inequality holds since $\sqrt{134} \le 5 + 7$.

   (b) $\mathbf{v} = (i, -i, 1+i)$ and $\mathbf{w} = (2-i, 4, -2i)$ in $\mathbb{C}^3$ with the standard inner product.

   - $\langle \mathbf{v}, \mathbf{w}\rangle = i\cdot(2-i) + (-i)\cdot 4 + (1+i)\cdot(-2i) = -3$, $||\mathbf{v}|| = \sqrt{i\cdot(-i) + (-i)\cdot i + (1+i)(1-i)} = 2$, $||\mathbf{w}|| = \sqrt{(2-i)(2+i) + 4\cdot 4 + (-2i)(2i)} = 5$, and $||\mathbf{v}+\mathbf{w}|| = \sqrt{2^2 + (4-i)(4+i) + (1-i)(1+i)} = \sqrt{23}$.
   - Cauchy-Schwarz holds since $|-3| \le 2\cdot 5$, and the triangle inequality holds since $\sqrt{23} \le 2 + 5$.

   (c) $\mathbf{v} = e^t$ and $\mathbf{w} = e^{2t}$ in $C[0,1]$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

   - We have $\langle \mathbf{v}, \mathbf{w}\rangle = \int_0^1 e^{3t}\,dt = \frac{1}{3}(e^3 - 1)$, $||\mathbf{v}|| = \sqrt{\int_0^1 e^{2t}\,dt} = \sqrt{\frac{1}{2}(e^{2t} - 1)}$, $||\mathbf{w}|| = \sqrt{\int_0^1 e^{4t}\,dt} = \sqrt{\frac{1}{4}(e^{4t} - 1)}$, and $||\mathbf{v}+\mathbf{w}|| = \sqrt{\int_0^1 (e^t + e^{2t})^2\,dt} = \sqrt{\frac{1}{4}e^4 + \frac{2}{3}e^3 + \frac{1}{2}e^2 - \frac{17}{12}}$.
   - Cauchy-Schwarz holds since $\langle \mathbf{v}, \mathbf{w}\rangle \approx 6.362$ while $||\mathbf{v}||\,||\mathbf{w}|| \approx 6.543$, and the triangle inequality holds $||\mathbf{v}+\mathbf{w}|| \approx 5.415$ while $||\mathbf{v}|| + ||\mathbf{w}|| \approx 5.448$.

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4. For each list S of vectors, apply Gram-Schmidt to calculate an orthogonal basis for span(S):

   (a) $\mathbf{v}_1 = (2,4,-4)$, $\mathbf{v}_2 = (1,-1,4)$, $\mathbf{v}_3 = (1,1,1)$ in $\mathbb{R}^3$ under the standard dot product.

   - We start with $\mathbf{w}_1 = \mathbf{v}_1 = \boxed{(2,4,-4)}$.
   - Next, $\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{w}_1$, where $a_1 = \dfrac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{(1,-1,4)\cdot(2,4,-4)}{(2,4,-4)\cdot(2,4,-4)} = \dfrac{-18}{36}$. Thus, $\mathbf{w}_2 = \boxed{(2,1,2)}$.
   - Finally, $\mathbf{w}_3 = \mathbf{v}_3 - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$ where $b_1 = \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{(1,1,1)\cdot(2,4,-4)}{(2,4,-4)\cdot(2,4,-4)} = \dfrac{2}{36}$, and $b_2 = \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} = \dfrac{(1,1,1)\cdot(2,1,2)}{(2,1,2)\cdot(2,1,2)} = \dfrac{5}{9}$. Thus, $\mathbf{w}_3 = \boxed{(-\frac{2}{9}, \frac{2}{9}, \frac{1}{9})}$.

   (b) $\mathbf{v}_1 = (1,2,0,-2)$, $\mathbf{v}_2 = (1,-1,4,4)$, $\mathbf{v}_3 = (6,6,0,-9)$ in $\mathbb{R}^4$ under the standard dot product.

   - First, $\mathbf{w}_1 = \mathbf{v}_1 = \boxed{(1,2,0,-2)}$.
   - Next, $\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{w}_1$, where $a_1 = \dfrac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{(1,-1,4,4)\cdot(1,2,0,-2)}{(1,2,0,-2)\cdot(1,2,0,-2)} = \dfrac{-9}{9} = -1$. Thus, $\mathbf{w}_2 = (1,-1,4,4) + (1,2,0,-2) = \boxed{(2,1,4,2)}$.
   - Finally, $\mathbf{w}_3 = \mathbf{v}_3 - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$ where $b_1 = \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{(6,6,0,-9)\cdot(1,2,0,-2)}{(1,2,0,-2)\cdot(1,2,0,-2)} = 4$, and $b_2 = \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} = \dfrac{(6,6,0,-9)\cdot(2,1,4,2)}{(2,1,4,2)\cdot(2,1,4,2)} = 0$. Thus, $\mathbf{w}_3 = (6,6,0,-9) - 4(1,2,0,-2) - 0(2,1,4,2) = \boxed{(2,-2,0,-1)}$.

   (c) $\mathbf{v}_1 = x$, $\mathbf{v}_2 = x^2$, $\mathbf{v}_3 = x^3$ in $C[-1,1]$ under the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

   - We start with $\mathbf{w}_1 = p_1 = \boxed{x}$.
   - Next, $\mathbf{w}_2 = p_2 - a_1\mathbf{w}_1$, where $a_1 = \dfrac{\langle p_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle} = \dfrac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} = 0$. Thus, $\mathbf{w}_2 = \boxed{x^2}$.
   - Finally, $\mathbf{w}_3 = p_3 - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$ where $b_1 = \dfrac{\langle p_3, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle} = \dfrac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx} = \dfrac{3}{5}$, and $b_2 = \dfrac{\langle p_3, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle} = \dfrac{\int_{-1}^{1} x^5\,dx}{\int_{-1}^{1} x^4\,dx} = 0$. Thus, $\mathbf{w}_3 = \boxed{x^3 - \frac{3}{5}x}$.

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5. Let F be a field and $n \geq 2$ be an integer. Recall that we say two matrices A and B in $M_{n\times n}(F)$ are <u>similar</u> when there exists an invertible $n \times n$ matrix with $B = Q^{-1}AQ$.

   (a) Show that if A and B are similar matrices in $M_{n\times n}(F)$, then $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. [Hint: You may use the fact that $\mathrm{tr}(CD) = \mathrm{tr}(DC)$.]

   - If $B = Q^{-1}AQ$, then $\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = \dfrac{1}{\det(Q)}\det(A)\det(Q) = \det(A)$.
   - Likewise, using the property in the hint with $C = Q^{-1}$ and $D = AQ$, we see $\mathrm{tr}(B) = \mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}(AQQ^{-1}) = \mathrm{tr}(A)$.

   (b) Show that "being similar" is an equivalence relation on $M_{n\times n}(F)$.

   - Reflexive: Every matrix is similar to itself, since $A = I_n^{-1}AI_n$.
   - Symmetric: If A is similar to B, so that $B = Q^{-1}AQ$, then $A = QBQ^{-1} = (Q^{-1})^{-1}BQ^{-1}$, so B is similar to A.
   - Transitive: If A is similar to B and B is similar to C, say with $A = Q^{-1}BQ$ and $B = R^{-1}CR$, then $A = R^{-1}Q^{-1}CQR = (QR)^{-1}C(QR)$, so A is similar to C.

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6. Let V be an inner product space.

   (a) If $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on V, show that $\langle \cdot, \cdot \rangle_3 = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ is also an inner product on V, where $\langle \mathbf{v}, \mathbf{w} \rangle_3 = \langle \mathbf{v}, \mathbf{w} \rangle_1 + \langle \mathbf{v}, \mathbf{w} \rangle_2$.

   - We verify the three requirements.
   - [I1]: $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle_3 = \langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle_1 + \langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle_2 = \langle \mathbf{x}, \mathbf{z} \rangle_1 + \langle \mathbf{y}, \mathbf{z} \rangle_1 + \langle \mathbf{x}, \mathbf{z} \rangle_2 + \langle \mathbf{y}, \mathbf{z} \rangle_2 = \langle \mathbf{x}, \mathbf{z} \rangle_3 + \langle \mathbf{y}, \mathbf{z} \rangle_3$.
   - [I2]: $\overline{\langle \mathbf{y}, \mathbf{x} \rangle_3} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle_1 + \langle \mathbf{y}, \mathbf{x} \rangle_2} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle_1} + \overline{\langle \mathbf{y}, \mathbf{x} \rangle_2} = \langle \mathbf{x}, \mathbf{y} \rangle_1 + \langle \mathbf{x}, \mathbf{y} \rangle_2 = \langle \mathbf{x}, \mathbf{y} \rangle_3$.
   - [I3]: $\langle \mathbf{x}, \mathbf{x} \rangle_3 = \langle \mathbf{x}, \mathbf{x} \rangle_1 + \langle \mathbf{x}, \mathbf{x} \rangle_2 \geq 0 + 0 = 0$, and equality can occur only when both terms are zero, which requires $\mathbf{x} = 0$.

   (b) If $\langle \cdot, \cdot \rangle_1$ is an inner product on V and c is a positive real number, show that $\langle \cdot, \cdot \rangle_3 = c \langle \cdot, \cdot \rangle_1$ is also an inner product on V, where $\langle \mathbf{v}, \mathbf{w} \rangle_3 = c \langle \mathbf{v}, \mathbf{w} \rangle_1$.

   - We verify the three requirements.
   - [I1]: $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle_3 = c \langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle_1 = c \langle \mathbf{x}, \mathbf{z} \rangle_1 + c \langle \mathbf{y}, \mathbf{z} \rangle_1 = \langle \mathbf{x}, \mathbf{z} \rangle_3 + \langle \mathbf{y}, \mathbf{z} \rangle_3$.
   - [I2]: $\overline{\langle \mathbf{y}, \mathbf{x} \rangle_3} = \overline{c \langle \mathbf{y}, \mathbf{x} \rangle_1} = c \overline{\langle \mathbf{y}, \mathbf{x} \rangle_1} = c \langle \mathbf{x}, \mathbf{y} \rangle_1 = \langle \mathbf{x}, \mathbf{y} \rangle_3$.
   - [I3]: $\langle \mathbf{x}, \mathbf{x} \rangle_3 = c \langle \mathbf{x}, \mathbf{x} \rangle_1 \geq 0$ since c is a positive real, and equality can occur only when $\langle \mathbf{x}, \mathbf{x} \rangle_1 = 0$, which requires $\mathbf{x} = 0$.

   (c) Does the collection of inner products on V form a vector space under the natural addition and scalar multiplication described above? Explain why or why not.

   - $\boxed{\text{No}}$: this collection is not closed under scaling by a negative scalar (or zero, or a non-real scalar if the underlying field is $\mathbb{C}$) so it is not a vector space.

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7. Prove the following inequalities:

   (a) Prove that $(a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) \geq n^2$ for any positive real numbers $a_1, a_2, \ldots, a_n$, with equality if and only if all of the a_i are equal.

   - Let $\mathbf{v} = (\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n})$ and $\mathbf{w} = \left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \ldots, \frac{1}{\sqrt{a_n}} \right)$.
   - We see that $\mathbf{v} \cdot \mathbf{w} = n$ while $||\mathbf{v}|| = \sqrt{a_1 + a_2 + \cdots + a_n}$ and $||\mathbf{w}|| = \sqrt{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$.
   - So applying the Cauchy-Schwarz inequality and squaring both sides produces the required $n^2 \leq (a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right)$.

   (b) If a, b, c, d are real numbers with $a^2 + b^2 + c^2 + d^2 \leq 5$, show that $a + 2b + 3c + 4d \leq 5\sqrt{6}$.

   - Let $\mathbf{v} = (1, 2, 3, 4)$ and $\mathbf{w} = (a, b, c, d)$. Then $\mathbf{v} \cdot \mathbf{w} = a + 2b + 3c + 4d$, $||\mathbf{v}||^2 = 1^2 + 2^2 + 3^2 + 4^2$, and $||\mathbf{w}||^2 = a^2 + b^2 + c^2 + d^2$.
   - Thus, by the Cauchy-Schwarz inequality, we have $(a + 2b + 3c + 4d)^2 \leq (1^2 + 2^2 + 3^2 + 4^2)(a^2 + b^2 + c^2 + d^2) = 150$.
   - Taking the square root gives $a + 2b + 3c + 4d \leq \sqrt{150} = 5\sqrt{6}$, as claimed.

   (c) Prove Nesbitt's inequality: for any positive real numbers a, b, c it is true that $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$. [Hint: Apply Cauchy-Schwarz to $(\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ and $(1/\sqrt{a+b}, 1/\sqrt{b+c}, 1/\sqrt{c+a})$.]

   - Following the hint, if $\mathbf{v} = (\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ and $\mathbf{w} = (1/\sqrt{a+b}, 1/\sqrt{b+c}, 1/\sqrt{c+a})$, then $\mathbf{v} \cdot \mathbf{w} = 3$, $||\mathbf{v}||^2 = (a+b) + (b+c) + (c+a) = 2(a+b+c)$, and $||\mathbf{w}||^2 = \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}$.
   - Thus, by the Cauchy-Schwarz inequality, we have $\frac{2(a+b+c)}{a+b} + \frac{2(a+b+c)}{b+c} + \frac{2(a+b+c)}{c+a} \geq 9$.
   - Dividing both sides by 3 and simplifying the fractions yields $\frac{c}{a+b} + 1 + \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 \geq \frac{9}{2}$, which immediately implies the desired inequality $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$.

(d) Prove the following generalization of Cauchy-Schwarz: if $\langle \cdot, \cdot \rangle$ is an inner product on the vector space V then $\left[\sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{w}_j \rangle\right]^2 \le \left[\sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{v}_j \rangle\right] \cdot \left[\sum_{j=1}^{n} \langle \mathbf{w}_j, \mathbf{w}_j \rangle\right]$ for any vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ in V. [Hint: Apply Cauchy-Schwarz to the space $\tilde{V}$ of n-tuples of elements of V, with an appropriate inner product on $\tilde{V}$.]

- Define $\tilde{V}$ to be the vector space of n-tuples of elements of V. Then for two elements $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_n)$, define the pairing $[\mathbf{x}, \mathbf{y}] = \sum_{j=1}^{n} \langle \mathbf{x}_j, \mathbf{y}_j \rangle$.
- This pairing is an inner product on $\tilde{V}$: [I1] and [I2] follow from their analogues for the inner product $\langle \cdot, \cdot \rangle$, while for [I3] we have $[\mathbf{x}, \mathbf{x}] = \sum_{j=1}^{n} \langle \mathbf{x}_j, \mathbf{x}_j \rangle = \sum_{j=1}^{n} ||\mathbf{x}_j||^2$ which is nonnegative and only equals zero when all of the $\mathbf{x}_j$ are zero.
- The desired inequality is then just the Cauchy-Schwarz inequality in $\tilde{V}$ applied to $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ and $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$.

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8. [Challenge] Let N be a positive integer and suppose we are given a set S_P of N points and a set S_L of N lines in the Euclidean plane. An "incidence" is defined to be a pair (P, L) with $P \in S_P$, $L \in S_L$, and where the point P lies on the line L. If I represents the total number of incidences, we have an obvious estimate $I \le N^2$; the goal of this problem is to prove a substantially better estimate of $I \le N^{3/2} + N$.

(a) Show that $I = \sum_{P \in S_P} \sum_{L \in S_L} \delta_{P,L}$ where $\delta_{P,L} = 1$ if P lies on L and 0 otherwise.

- This is immediate, since we are just summing the number 1 over all pairs (P, L) where P lies on L, which simply counts the number of incidences.

(b) For any fixed lines $L_1 \ne L_2$, show that $\sum_{P \in S_P} \delta_{P,L_1} \delta_{P,L_2} \le 1$.

- Note that $\delta_{P,L_1} \delta_{P,L_2} = 1$ only when the point P lies on both lines L_1 and L_2, and is 0 otherwise.
- Therefore, when $L_1 \ne L_2$, the sum is at most 1 (and in fact equals 1 only when L_1 intersects L_2 at one of the points).

(c) Show that $\sum_{P \in S_P} (\sum_{L \in S_L} \delta_{P,L})^2 \le I + (N^2 - N)$. [Hint: Write $(\sum_{L \in S_L} \delta_{P,L})^2 = (\sum_{L_1 \in S_L} \delta_{P,L_1})(\sum_{L_2 \in S_L} \delta_{P,L_2})$ and then split apart into the terms where $L_1 = L_2$ and where $L_1 \ne L_2$.]

- Per the hint we first observe that $(\sum_{L \in S_L} \delta_{P,L})^2 = (\sum_{L_1 \in S_L} \delta_{P,L_1})(\sum_{L_2 \in S_L} \delta_{P,L_2}) = \sum_{L_1, L_2 \in S_L} \delta_{P,L_1} \delta_{P,L_2}$.
- Now splitting this sum into the terms where $L_1 = L_2$ and where $L_1 \ne L_2$ we see that $(\sum_{L \in S_L} \delta_{P,L})^2 = \sum_{L \in S_L} \delta_{P,L} + \sum_{L_1, L_2 \in S_L, L_1 \ne L_2} \delta_{P,L_1} \delta_{P,L_2}$.
- Summing over P yields $\sum_{P \in S_P} (\sum_{L \in S_L} \delta_{P,L})^2 = \sum_{P \in S_P} \sum_{L \in S_L} \delta_{P,L} + \sum_{L_1, L_2 \in S_L, L_1 \ne L_2} \sum_{P \in S_P} \delta_{P,L_1} \delta_{P,L_2}$.
- The first sum is simply I by part (a), while the second sum is $\le \sum_{L_1, L_2 \in S_L, L_1 \ne L_2} = N^2 - N$ by part (b) and the fact that there are $N(N-1)$ pairs of distinct lines (L_1, L_2) in the set.

(d) Show that $I^2 \le IN + N(N^2 - N)$ and deduce that $I \le N^{3/2} + N$. [Hint: Use Cauchy-Schwarz on the outer sum in (a).]

- By Cauchy-Schwarz we have $I^2 = [\sum_{P \in S_P} \sum_{L \in S_L} \delta_{P,L}]^2 \le [\sum_{P \in S_P} 1][\sum_{P \in S_P} (\sum_{L \in S_L} \delta_{P,L})^2] \le N \cdot (I + (N^2 - N))$.
- Completing the square we see $(I - N/2)^2 \le N^3 - N^2 + N^2/4 < (N^{3/2})^2$ hence $I - N/2 \le N^{3/2}$ so $I \le N^{3/2} + N$, as claimed. (In fact we get a slightly better estimate $I \le N^{3/2} + N/2$.)

Remark: This technique as used in algebraic combinatorics is often called the "L^2 method", and there are many open questions related to this one seeking optimal estimates and constructions for point-line incidences.
