

1. Assume that the vector spaces U, V, W are finite-dimensional over the field F , the bases $\alpha, \beta, \gamma, \delta$ are ordered, and that S, T are linear transformations. Identify each of the following statements as true or false:

- (a) If $T : V \rightarrow W$ and $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$, then T is one-to-one.
- **True**: this is a result we established in class. In fact, it is an if-and-only if.
- (b) If $T : V \rightarrow W$ is one-to-one, then $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.
- **True**: this is the converse of (a), and is also a result we established in class.
- (c) If $T : V \rightarrow W$ and S spans V , then $T(S) = \{T(\mathbf{s}) : \mathbf{s} \in S\}$ spans W .
- **False**: we showed $T(S)$ is a spanning set for $\text{im}(T)$, but if T is not onto then $T(S)$ cannot span W .
- (d) If $T : V \rightarrow W$ and S is linearly independent in V , then $T(S)$ is a linearly independent in W .
- **False**: if T is one-to-one then T preserves independence, but arbitrary T need not (e.g., $T = 0$).
- (e) If $T : V \rightarrow W$ and S is a basis for V , then $T(S)$ is a basis for W .
- **False**: as in (d), $T(S)$ need not be linearly independent nor span W .
- (f) If $T : V \rightarrow W$ and $\dim(V) = \dim(W)$, then T is an isomorphism.
- **False**: there are lots of counterexamples, such as having T be the zero map.
- (g) If $T : V \rightarrow W$ and for any $\mathbf{w} \in W$ there is a unique $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$, then T is an isomorphism.
- **True**: this statement is equivalent to saying that T has an inverse function $T^{-1} : W \rightarrow V$, which is the same as saying that T is an isomorphism.
- (h) If V is isomorphic to W , then $\dim(V) = \dim(W)$.
- **True**: an isomorphism maps a basis of V to a basis of W , so $\dim(V) = \dim(W)$.
- (i) The space $\mathcal{L}(V, W)$ of all linear transformations from V to W has dimension $\dim V \cdot \dim W$.
- **True**: since $\mathcal{L}(V, W)$ is isomorphic to $M_{\dim(V) \times \dim(W)}(F)$, their dimensions are also equal.
- (j) If A is an $m \times n$ matrix of rank r , then the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension r .
- **False**: the solution space is the nullspace, which has dimension $n - r$ if the matrix has rank r .
- (k) If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_{\beta}^{\gamma}$ is an element of $M_{m \times n}(F)$.
- **False**: if $\dim(V) = m$ and $\dim(W) = n$ then $[T]_{\beta}^{\gamma}$ is an $n \times m$ matrix. (Try it!)
- (l) If $[S]_{\alpha}^{\beta} = [T]_{\alpha}^{\beta}$ then $S = T$.
- **True**: the map associating a linear transformation with its associated matrix is an isomorphism, so two linear transformations have the same associated matrix if and only if they are equal.
- (m) If $[T]_{\alpha}^{\beta} = [T]_{\gamma}^{\delta}$ then $\alpha = \gamma$ and $\beta = \delta$.
- **False**: for example if T is the identity map on \mathbb{R}^2 and α, β are the standard basis and γ, δ are twice the standard basis, the associated matrices are both the identity matrix but the bases are different.
- (n) If $S : V \rightarrow W$ and $T : V \rightarrow W$ then $[S + T]_{\alpha}^{\beta} = [S]_{\alpha}^{\beta} + [T]_{\alpha}^{\beta}$.
- **True**: this is the correct rule for computing the matrix associated to a sum.
- (o) If $T : V \rightarrow W$ and $\mathbf{v} \in V$, then $[T]_{\alpha}^{\beta}[\mathbf{v}]_{\beta} = [T\mathbf{v}]_{\alpha}$.
- **False**: the correct formula is $[T]_{\alpha}^{\beta}[\mathbf{v}]_{\alpha} = [T\mathbf{v}]_{\beta}$.
- (p) If $S : V \rightarrow W$ and $T : U \rightarrow V$, then $[ST]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$.
- **True**: this is the correct rule for computing the matrix associated to a composition.

2. For each linear transformation T and given bases β and γ , find $[T]_{\beta}^{\gamma}$:

(a) $T : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ given by $T(a, b) = \langle a - b, b - 2a, 3b \rangle$, with $\beta = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$, $\gamma = \{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$.

• We have $T(1, 0) = \langle 1, -2, 0 \rangle$ and $T(0, 1) = \langle -1, 1, 3 \rangle$ so the matrix is $[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 0 & 3 \end{bmatrix}$.

(b) The trace map from $M_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$ and $\gamma = \{1\}$.

• We have $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1$, $T\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) = 0$, $T\left(\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\right) = 0$, and $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 5$.

• So the matrix is $[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 0 & 0 & 5 \end{bmatrix}$.

(c) $T : \mathbb{Q}^4 \rightarrow P_4(\mathbb{Q})$ given by $T(a, b, c, d) = a + (a + b)x + (a + 3c)x^2 + (2a + d)x^3 + (b + 5c + d)x^4$, with β the standard basis and $\gamma = \{x^3, x^2, x^4, x, 1\}$.

• We have $T(1, 0, 0, 0) = 1 + x + x^2 + 2x^3$, $T(0, 1, 0, 0) = x + x^4$, $T(0, 0, 1, 0) = 3x^2 + 5x^4$, and

$T(0, 0, 0, 1) = x^3 + x^4$. Thus, $[T]_{\beta}^{\gamma} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(d) $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ given by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$ with $\beta = \gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

• We have $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$, and

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$. Thus, $[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$.

(e) The matrix $[T]_{\beta}^{\gamma}$ associated to the linear transformation $T : P_3(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ with $T(p) = x^2 p'(x)$, where $\beta = \{1 - x, 1 - x^2, 1 - x^3, x^2 + x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$.

• Since $T(1 - x) = -x^2$, $T(1 - x^2) = -2x^3$, $T(1 - x^3) = -3x^4$, $T(x^2 + x^3) = 2x^3 + 3x^4$, the matrix is

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & 3 \end{bmatrix}$.

(f) The projection map (see problem 6 of homework 4) on \mathbb{R}^3 that maps the vectors $\langle 1, 2, 1 \rangle$ and $\langle 0, -3, 1 \rangle$ to themselves and sends $\langle 1, 1, 1 \rangle$ to the zero vector, with $\beta = \gamma = \{\langle 1, 2, 1 \rangle, \langle 0, -3, 1 \rangle, \langle 1, 1, 1 \rangle\}$.

• We have $T(1, 2, 1) = \langle 1, 2, 1 \rangle$, $T(0, -3, 1) = \langle 0, -3, 1 \rangle$, and $T(1, 1, 1) = \langle 0, 0, 0 \rangle$.

• So the matrix is simply $[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(g) The same map as in part (f), but relative to the standard basis for \mathbb{R}^3 .

• One approach is to compute the action of T on the standard basis directly. Another approach is to use the change-of-basis formula: if α is the standard basis and β is the basis from (f), then $Q = [T]_{\beta}^{\alpha} =$

$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ so $[T]_{\alpha}^{\beta} = Q^{-1} = \begin{bmatrix} -4 & 1 & 3 \\ -1 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix}$. Then $[T]_{\alpha}^{\alpha} = Q[T]_{\beta}^{\beta}Q^{-1} = \begin{bmatrix} -4 & 1 & 3 \\ -5 & 2 & 3 \\ -5 & 1 & 4 \end{bmatrix}$.

3. Let $T : P_3(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ be given by $T(p) = x^2 p''(x)$.

(a) With the bases $\alpha = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$, find $[T]_\alpha^\gamma$.

• We have $T(1) = 0$, $T(x) = 0$, $T(x^2) = 2x^2$, and $T(x^3) = 6x^3$, so $[T]_\alpha^\gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) If $q(x) = 1 - x^2 + 2x^3$, compute $[q]_\alpha$ and $[T(q)]_\gamma$ and verify that $[T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha$.

• We have $[q]_\alpha = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ and $T(q) = -2x^2 + 12x^3$, so $[T(q)]_\gamma = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 12 \\ 0 \end{bmatrix}$. Indeed, $[T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha$.

Notice that $T = SU$ where $U : P_3(\mathbb{R}) \rightarrow P_1(\mathbb{R})$ has $U(p) = p''(x)$ and $S : P_1(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ has $S(p) = x^2 p(x)$.

(c) With $\beta = \{1, x\}$, compute the associated matrices $[S]_\beta^\gamma$, and $[U]_\alpha^\beta$ and then verify that $[T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\alpha^\beta$.

• Since $S(1) = x^2$ and $S(x) = x^3$ we have $[S]_\beta^\gamma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and likewise since $U(1) = 0$, $U(x) = 0$, $U(x^2) = 2$, and $U(x^3) = 6x$, we see $[U]_\alpha^\beta = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$. Then $[T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\alpha^\beta$ as claimed.

(d) Which of S , T , and U are onto? One-to-one? Isomorphisms?

- The map U is onto (since its image is all of $P_1(\mathbb{R})$), but S and T are not onto.
- Also, S is one-to-one, since its kernel is trivial, but U and T both have nonzero elements in their kernels, so they are not one-to-one.
- Since none of the maps is both one-to-one and onto, none of them are isomorphisms.

4. Suppose that $T : V \rightarrow W$ is a linear transformation.

(a) If T is onto, show that $\dim(W) \leq \dim(V)$.

- If T is onto, then $\text{im} T = W$, so $\dim(\ker T) + \dim(W) = \dim(V)$. Since $\dim(\ker T) \geq 0$, this means $\dim(W) \leq \dim(V)$.

(b) If T is one-to-one, show that T is an isomorphism from V to $\text{im}(T)$, and deduce that $\dim(V) \leq \dim(W)$.

- For the first part, if T is one-to-one, then $T : V \rightarrow \text{im}(T)$ is a one-to-one map that is onto (by definition of $\text{im} T$), meaning it is an isomorphism.
- For the second part, since T is an isomorphism from V to $\text{im}(T)$, we have $\dim(\text{im} T) = \dim(V)$. But since $\text{im}(T)$ is a subspace of W , we see that $\dim(\text{im} T) \leq \dim(W)$, so $\dim(V) = \dim(\text{im} T) \leq \dim(W)$.

5. Let V be a vector space and $T : V \rightarrow V$ be linear.

(a) If V is finite-dimensional and $\ker(T) \cap \text{im}(T) = \{\mathbf{0}\}$, prove in fact that $V = \ker(T) \oplus \text{im}(T)$. [Hint: Use problem 6 from homework 3.]

- Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\ker(T)$ and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a basis for $\text{im}(T)$.
- Since $\ker(T) \cap \text{im}(T) = \{\mathbf{0}\}$, by problem 6(a) of homework 3, we see that the union $\beta \cup \gamma$ is a basis for $\ker(T) + \text{im}(T)$.

- But then by the nullity-rank theorem, $\ker(T) + \text{im}(T)$ has dimension $m + n = \dim(\ker T) + \dim(\text{im} T) = \dim(V)$, and so $\ker(T) + \text{im}(T) = V$.
 - This means $V = \ker(T) \oplus \text{im}(T)$, as claimed.
- (b) Give a counterexample showing the result of (a) can be false if V is infinite-dimensional.
- There are various possible counterexamples.
 - One possibility is the right-shift map $R(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots)$ whose kernel is zero but which is not onto: so $\ker(R) \cap \text{im}(R) = \{\mathbf{0}\}$ but $V \neq \ker(R) + \text{im}(R)$.
 - Another is the antiderivative map on polynomials: $A(p) = \int_0^x p(t) dt$. Again, the kernel is zero but the map is not onto (since the image contains no nonzero constants).
- (c) If V is finite-dimensional and $V = \ker(T) + \text{im}(T)$, prove in fact that $V = \ker(T) \oplus \text{im}(T)$.
- Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\ker(T)$ and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a basis for $\text{im}(T)$.
 - Since $V = \ker(T) + \text{im}(T)$ has dimension $\dim(V) = \dim(\ker T) + \dim(\text{im} T) = m + n$, since $\beta \cup \gamma$ spans V it must be linearly independent. Then by problem 6(b) of homework 3, we see that $\ker(T) \cap \text{im}(T) = \{\mathbf{0}\}$.
 - This means by problem 6(a) of homework 3, we see that the union $\beta \cup \gamma$ is a basis for $\ker(T) + \text{im}(T)$.
 - This means $V = \ker(T) \oplus \text{im}(T)$, as claimed.
- (d) Give a counterexample showing the result of (c) can be false if V is infinite-dimensional.
- There are various possible counterexamples.
 - One possibility is the left-shift map $L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$ whose kernel is the sequences $(a_1, 0, 0, \dots)$ but which is also onto: so $V = \ker(T) + \text{im}(T)$ but $\ker(T) \cap \text{im}(T) \neq \{\mathbf{0}\}$.
 - Another is the derivative map on polynomials: $A(p) = p'$. Again, the kernel is nonzero but the map is onto.

6. Let F be a field and d be a positive integer.

- (a) Show that any polynomial in $P_d(F)$ with more than d distinct roots must be the zero polynomial. [Hint: Use the factor theorem.]
- Suppose $p(x)$ is a polynomial of degree at most d with $d + 1$ distinct roots r_1, \dots, r_{d+1} .
 - By the factor theorem, $p(x)$ is divisible by $x - r_1$, say $p(x) = (x - r_1)p_1(x)$ where p_1 has degree $d - 1$.
 - Then since $0 = p(r_2) = (r_2 - r_1)p_1(r_2)$ we must have p_1 divisible by $x - r_2$, so $p_1(x) = (x - r_2)p_2(x)$ where p_2 has degree $d - 2$, meaning $p(x) = (x - r_1)(x - r_2)p_2(x)$.
 - Iterating this procedure (or by a trivial induction) shows that $p(x) = (x - r_1)(x - r_2) \cdots (x - r_d)p_d(x)$ for some polynomial $p_d(x)$ of degree 0. But setting $x = r_{d+1}$ shows that $p_d(r_{d+1}) = 0$ so since p_d is constant, it must be zero. Then $p(x) = 0$ is the zero polynomial.

Now let a_0, a_1, \dots, a_d be distinct elements of F and consider the linear transformation $T : P_d(F) \rightarrow F^{d+1}$ given by $T(p) = (p(a_0), p(a_1), \dots, p(a_d))$.

- (b) Show that $\ker(T) = \{\mathbf{0}\}$ and deduce that T is an isomorphism.
- If p is in $\ker(T)$, then $p(a_0) = p(a_1) = \dots = p(a_d) = 0$, meaning that p has $d + 1$ roots. Thus, by (a), p must be the zero polynomial. Thus $\ker(T) = \{\mathbf{0}\}$ as desired.
 - Then by the nullity-rank theorem, we have $\dim(\text{im} T) = d + 1 = \dim(F^{d+1})$, so T is also onto, and is therefore an isomorphism.
- (c) Conclude that, for any list of $d + 1$ points $(a_0, b_0), \dots, (a_d, b_d)$ with distinct first coordinates, there exists a unique polynomial of degree at most d having the property that $p(a_i) = b_i$ for each $0 \leq i \leq d$.
- This is (ultimately) just a restatement of the fact that T is an isomorphism from (b): since T is onto, there is a polynomial $p(x)$ in $P_d(F)$ with $T(p) = (b_0, b_1, \dots, b_d)$, which is the same as saying that $p(a_0) = b_0, \dots, p(a_d) = b_d$.
 - Furthermore, since T is one-to-one, there is only one such polynomial.

7. Suppose that V is a finite-dimensional vector space and $T : V \rightarrow V$ is linear.

(a) Suppose there exists a basis β of V such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. Show that T is a projection map (i.e., that $T^2 = T$).

- The product of two diagonal matrices is the diagonal matrix whose diagonal entries are just the products of the corresponding entries of the original matrices: explicitly, we have

$$\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & & & \\ & a_2 b_2 & & \\ & & \ddots & \\ & & & a_n b_n \end{bmatrix}.$$

- So if M is diagonal with diagonal entries all 1s and 0s, then since $1^2 = 1$ and $0^2 = 0$ we immediately see that $M^2 = M$.
- Then $[T^2]_\beta^\beta = [T]_\beta^\beta [T]_\beta^\beta = [T]_\beta^\beta$, so T^2 and T have the same associated matrix, hence are equal as linear transformations.

(b) Conversely, suppose that T is a projection map. Show that there exists a basis β of V such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. [Hint: As shown on homework 4, $V = \ker(T) \oplus \text{im}(T)$; take β be a basis of $\ker(T)$ followed by a basis of $\text{im}(T)$.]

- Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a basis of $\ker(T)$ followed by a basis of $\text{im}(T)$ where the $\mathbf{v}_i \in \ker(T)$ and the $\mathbf{w}_i \in \text{im}(T)$.
- Then $T(\mathbf{v}_i) = \mathbf{0} = 0\mathbf{v}_i$, and also for any $\mathbf{w} = T(\mathbf{v})$ in $\text{im}(T)$ we have $T(\mathbf{w}) = T(T(\mathbf{v})) = T(\mathbf{v}) = \mathbf{w}$, so $T(\mathbf{w}_i) = \mathbf{w}_i = 1\mathbf{w}_i$ for each i . (The fact that T acts as the identity on $\text{im}(T)$ was also shown in problem 8 of homework 4.)
- Thus, the associated matrix $[T]_\beta^\beta$ is the diagonal matrix whose first n diagonal entries are 0s and whose remaining m diagonal entries are 1s, which has the desired form.

8. [Challenge] The goal of this problem is to discuss dual vector spaces. If V is an F -vector space, its dual space V^* is the set of F -valued linear transformations $T : V \rightarrow F$. Observe that V^* is a vector space under pointwise addition and scalar multiplication.

If $\beta = \{\mathbf{e}_i\}_i$ is a basis of V , its associated dual set is the set $\beta^* = \{e_i^*\}_i$ where $e_i^* : V \rightarrow F$ is defined by $e_i^*(\mathbf{e}_i) = 1$ and $e_i^*(\mathbf{e}_j) = 0$ for $i \neq j$. (In other words, e_i^* is the linear transformation that sends \mathbf{e}_i to 1 and all of the other basis vectors in β to 0.)

(a) Show that the dual set β^* is linearly independent.

- Suppose we have a linear dependence $a_1 e_1^* + a_2 e_2^* + \dots + a_n e_n^* = 0$, meaning that this function $a_1 e_1^* + a_2 e_2^* + \dots + a_n e_n^*$ evaluates to zero on every vector in V .
- In particular, evaluating this function on the vector \mathbf{e}_i yields the coefficient a_i . Hence $a_i = 0$ for all i .

(b) If V is finite-dimensional, let $f \in V^*$. Show that $f = \sum_i f(\mathbf{e}_i) e_i^*$. Deduce that the dual set β^* is a basis of V^* and that $\dim(V^*) = \dim(V)$. [Hint: Show f agrees with the sum on each \mathbf{e}_j .]

- Let $g = \sum_i f(\mathbf{e}_i) e_i^*$. Then $g(\mathbf{e}_j) = [\sum_i f(\mathbf{e}_i) e_i^*](\mathbf{e}_j) = \sum_i f(\mathbf{e}_i) \cdot e_i^*(\mathbf{e}_j)$, but $e_i^*(\mathbf{e}_j) = 0$ except when $i = j$. So the sum reduces just to the term where $i = j$: namely, $g(\mathbf{e}_j) = f(\mathbf{e}_j) \cdot e_j^*(\mathbf{e}_j) = f(\mathbf{e}_j)$.
- Therefore, we see that g and f agree on each basis vector \mathbf{e}_j , so since a linear transformation is characterized by its values on a basis, g and f are equal as functions.
- The second part follows immediately, since it implies that the $\{e_i^*\}_i$ span V^* , so by part (a), they are a basis for V^* . For the last part we simply observe that there are the same number of e_i^* as \mathbf{e}_i , so $\dim(V^*) = \dim(V)$.

Part (b) shows that when V is finite-dimensional, the association $\{\mathbf{e}_i\}_i \rightarrow \{e_i^*\}_i$ extends to an isomorphism of V with V^* . However, this isomorphism depends on a choice of a specific basis for V . Iterating this map shows that V is also isomorphic with its double-dual V^{**} : interestingly, however, there exists a natural isomorphism of V with V^{**} that does not require a specific choice of basis.

- (c) For $\mathbf{v} \in V$, define the “evaluation-at- \mathbf{v} map” $\hat{\mathbf{v}} : V^* \rightarrow F$ by setting $\hat{\mathbf{v}}(f) = f(\mathbf{v})$ for every $f \in V^*$: then $\hat{\mathbf{v}}$ is an element of V^{**} . When V is finite-dimensional, show that the map $\varphi : V \rightarrow V^{**}$ with $\varphi(\mathbf{v}) = \hat{\mathbf{v}}$ is an isomorphism. [Hint: Show φ is linear and one-to-one.]
- First, φ is linear: for $\mathbf{v}, \mathbf{w} \in V$ and $f \in V^*$ we have $\varphi(\mathbf{v} + \alpha\mathbf{w})(f) = f(\mathbf{v} + \alpha\mathbf{w}) = f(\mathbf{v}) + \alpha f(\mathbf{w}) = \varphi(\mathbf{v})(f) + \alpha\varphi(\mathbf{w})(f)$.
 - Second, φ is one-to-one: suppose $\mathbf{v} \in V$ is nonzero. Then we may extend it to a basis β and take $f : V \rightarrow F$ to be the linear transformation mapping \mathbf{v} to 1 and the rest of β to 0: this means $f(\mathbf{v}) = 1$ and thus $\hat{\mathbf{v}}(f) = 1$, meaning that $\varphi(\hat{\mathbf{v}})(f) = 1$, so $\varphi(\hat{\mathbf{v}})$ is not zero.
 - Finally, since $\dim(V) = \dim(V^{**})$, since φ is one-to-one and linear, it is an isomorphism.

Essentially all of the results of (b) and (c) fail when V is infinite-dimensional.

- (d) For $V = F[x]$ with basis $\beta = \{1, x, x^2, x^3, \dots\}$, show that the linear transformation T with $T(p) = p(1)$ is not in $\text{span}(\beta^*)$. Deduce that β^* is not a basis of V^* .
- Note that the element e_i of the dual set β^* evaluates to 1 on x^i and 0 on other powers of x .
 - The linear transformation T , on the other hand, evaluates to 1 on all powers of x . It therefore cannot be written as a finite linear combination of the elements e_i^* , since any such element $a_0e_0^* + \dots + a_n e_n^*$ evaluates to zero on x^{n+1} .
 - Therefore, T is not in $\text{span}(\beta^*)$, so β^* does not span V^* hence is not a basis.

Remark: In part (d), it can in fact be shown that the dimension of V^* is uncountable, while the dimension of V is countable, so V^* and V are not even isomorphic.
