E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 5, due Tue Feb 10th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that the vector spaces $U, V, W$ are finite-dimensional over the field $F$, the bases $\alpha, \beta, \gamma, \delta$ are ordered, and that $S, T$ are linear transformations. Identify each of the following statements as true or false:

   (a) If $T : V \to W$ and $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$, then $T$ is one-to-one.
   (b) If $T : V \to W$ is one-to-one, then $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.
   (c) If $T : V \to W$ and $S$ spans $V$, then $T(S) = \{T(\mathbf{s}) : \mathbf{s} \in S\}$ spans $W$.
   (d) If $T : V \to W$ and $S$ is linearly independent in $V$, then $T(S)$ is a linearly independent in $W$.
   (e) If $T : V \to W$ and $S$ is a basis for $V$, then $T(S)$ is a basis for $W$.
   (f) If $T : V \to W$ and $\dim(V) = \dim(W)$, then $T$ is an isomorphism.
   (g) If $T : V \to W$ and for any $\mathbf{w} \in W$ there is a unique $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$, then $T$ is an isomorphism.
   (h) If $V$ is isomorphic to $W$, then $\dim(V) = \dim(W)$.
   (i) The space $\mathcal{L}(V, W)$ of all linear transformations from $V$ to $W$ has dimension $\dim V \cdot \dim W$.
   (j) If $A$ is an $m \times n$ matrix of rank $r$, then the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension $r$.
   (k) If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_\beta^\gamma$ is an element of $M_{m\times n}(F)$.
   (l) If $[S]_\alpha^\beta = [T]_\alpha^\beta$ then $S = T$.
   (m) If $[T]_\alpha^\beta = [T]_\gamma^\delta$ then $\alpha = \gamma$ and $\beta = \delta$.
   (n) If $S : V \to W$ and $T : V \to W$ then $[S + T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$.
   (o) If $T : V \to W$ and $\mathbf{v} \in V$, then $[T]_\alpha^\beta [\mathbf{v}]_\beta = [T\mathbf{v}]_\alpha$.
   (p) If $S : V \to W$ and $T : U \to V$, then $[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$.

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2. For each linear transformation $T$ and given bases $\beta$ and $\gamma$, find $[T]_\beta^\gamma$:

   (a) $T : \mathbb{C}^2 \to \mathbb{C}^3$ given by $T(a, b) = \langle a - b, b - 2a, 3b\rangle$, with $\beta = \{\langle 1, 0\rangle, \langle 0, 1\rangle\}$, $\gamma = \{\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\}$.
   (b) The trace map from $M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$ and $\gamma = \{1\}$.
   (c) $T : \mathbb{Q}^4 \to P_4(\mathbb{Q})$ given by $T(a, b, c, d) = a + (a + b)x + (a + 3c)x^2 + (2a + d)x^3 + (b + 5c + d)x^4$, with $\beta$ the standard basis and $\gamma = \{x^3, x^2, x^4, x, 1\}$.
   (d) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$ with $\beta = \gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.
   (e) The matrix $[T]_\beta^\gamma$ associated to the linear transformation $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ with $T(p) = x^2 p'(x)$, where $\beta = \{1 - x, 1 - x^2, 1 - x^3, x^2 + x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$.
   (f) The projection map (see problem 6 of homework 4) on $\mathbb{R}^3$ that maps the vectors $\langle 1, 2, 1\rangle$ and $\langle 0, -3, 1\rangle$ to themselves and sends $\langle 1, 1, 1\rangle$ to the zero vector, with $\beta = \gamma = \{\langle 1, 2, 1\rangle, \langle 0, -3, 1\rangle, \langle 1, 1, 1\rangle\}$.
   (g) The same map as in part (f), but relative to the standard basis for $\mathbb{R}^3$.

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3. Let $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ be given by $T(p) = x^2 p''(x)$.

   (a) With the bases $\alpha = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$, find $[T]_\alpha^\gamma$.

   (b) If $q(x) = 1 - x^2 + 2x^3$, compute $[q]_\alpha$ and $[T(q)]_\gamma$ and verify that $[T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha$.

   Notice that $T = SU$ where $U : P_3(\mathbb{R}) \to P_1(\mathbb{R})$ has $U(p) = p''(x)$ and $S : P_1(\mathbb{R}) \to P_4(\mathbb{R})$ has $S(p) = x^2 p(x)$.

   (c) With $\beta = \{1, x\}$, compute the associated matrices $[S]_\beta^\gamma$, and $[U]_\alpha^\beta$ and then verify that $[T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\alpha^\beta$.

   (d) Which of $S$, $T$, and $U$ are onto? One-to-one? Isomorphisms?

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose that $T : V \to W$ is a linear transformation.

   (a) If $T$ is onto, show that $\dim(W) \leq \dim(V)$.

   (b) If $T$ is one-to-one, show that $T$ is an isomorphism from $V$ to $\mathrm{im}(T)$, and deduce that $\dim(V) \leq \dim(W)$.

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5. Let $V$ be a vector space and $T : V \to V$ be linear.

   (a) If $V$ is finite-dimensional and $\ker(T) \cap \mathrm{im}(T) = \{\mathbf{0}\}$, prove in fact that $V = \ker(T) \oplus \mathrm{im}(T)$. [Hint: Use problem 6 from homework 3.]

   (b) Give a counterexample showing the result of (a) can be false if $V$ is infinite-dimensional.

   (c) If $V$ is finite-dimensional and $V = \ker(T) + \mathrm{im}(T)$, prove in fact that $V = \ker(T) \oplus \mathrm{im}(T)$.

   (d) Give a counterexample showing the result of (c) can be false if $V$ is infinite-dimensional.

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6. Let $F$ be a field and $d$ be a positive integer.

   (a) Show that any polynomial in $P_d(F)$ with more than $d$ distinct roots must be the zero polynomial. [Hint: Use the factor theorem.]

   Now let $a_0, a_1, \ldots, a_d$ be distinct elements of $F$ and consider the linear transformation $T : P_d(F) \to F^{d+1}$ given by $T(p) = (p(a_0), p(a_1), \ldots, p(a_d))$.

   (b) Show that $\ker(T) = \{\mathbf{0}\}$ and deduce that $T$ is an isomorphism.

   (c) Conclude that, for any list of $d+1$ points $(a_0, b_0), \ldots, (a_d, b_d)$ with distinct first coordinates, there exists a unique polynomial of degree at most $d$ having the property that $p(a_i) = b_i$ for each $0 \leq i \leq d$.

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7. Suppose that $V$ is a finite-dimensional vector space and $T : V \to V$ is linear.

   (a) Suppose there exists a basis $\beta$ of $V$ such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. Show that $T$ is a projection map (i.e., that $T^2 = T$).

   (b) Conversely, suppose that $T$ is a projection map. Show that there exists a basis $\beta$ of $V$ such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. [Hint: As shown on homework 4, $V = \ker(T) \oplus \mathrm{im}(T)$; take $\beta$ be a basis of $\ker(T)$ followed by a basis of $\mathrm{im}(T)$.]

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8. [Challenge] The goal of this problem is to discuss dual vector spaces. If $V$ is an $F$-vector space, its dual space $V^*$ is the set of $F$-valued linear transformations $T : V \to F$. Observe that $V^*$ is a vector space under pointwise addition and scalar multiplication.

   If $\beta = \{\mathbf{e}_i\}_i$ is a basis of $V$, its associated dual set is the set $\beta^* = \{e_i^*\}_i$ where $e_i^* : V \to F$ is defined by $e_i^*(\mathbf{e}_i) = 1$ and $e_i^*(\mathbf{e}_j) = 0$ for $i \neq j$. (In other words, $e_i^*$ is the linear transformation that sends $\mathbf{e}_i$ to 1 and all of the other basis vectors in $\beta$ to 0.)

   (a) Show that the dual set $\beta^*$ is linearly independent.

   (b) If $V$ is finite-dimensional, let $f \in V^*$. Show that $f = \sum_i f(\mathbf{e}_i)e_i^*$. Deduce that the dual set $\beta^*$ is a basis of $V^*$ and that $\dim(V^*) = \dim(V)$. [Hint: Show $f$ agrees with the sum on each $\mathbf{e}_j$.]

   Part (b) shows that when $V$ is finite-dimensional, the association $\{\mathbf{e}_i\}_i \to \{e_i^*\}_i$ extends to an isomorphism of $V$ with $V^*$. However, this isomorphism depends on a choice of a specific basis for $V$. Iterating this map shows that $V$ is also isomorphic with its double-dual $V^{**}$: interestingly, however, there exists a natural isomorphism of $V$ with $V^{**}$ that does not require a specific choice of basis.

   (c) For $\mathbf{v} \in V$, define the "evaluation-at-$\mathbf{v}$ map" $\hat{\mathbf{v}} : V^* \to F$ by setting $\hat{\mathbf{v}}(f) = f(\mathbf{v})$ for every $f \in V^*$: then $\hat{\mathbf{v}}$ is an element of $V^{**}$. When $V$ is finite-dimensional, show that the map $\varphi : V \to V^{**}$ with $\varphi(\mathbf{v}) = \hat{\mathbf{v}}$ is an isomorphism. [Hint: Show $\varphi$ is linear and one-to-one.]

   Essentially all of the results of (b) and (c) fail when $V$ is infinite-dimensional.

   (d) For $V = F[x]$ with basis $\beta = \{1, x, x^2, x^3, \dots\}$, show that the linear transformation $T$ with $T(p) = p(1)$ is not in $\text{span}(\beta^*)$. Deduce that $\beta^*$ is not a basis of $V^*$.

   **Remark:** In part (d), it can in fact be shown that the dimension of $V^*$ is uncountable, while the dimension of $V$ is countable, so $V^*$ and $V$ are not even isomorphic.

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