E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 4, due Tue Feb 3rd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false, where V and W are vector spaces:

   (a) If $\dim(V) = 5$, then there exists a set of 5 vectors in V that span V but are not linearly independent.

   (b) If $\dim(V) = 5$, then a set of 4 vectors in V cannot span V.

   (c) If $\dim(V) = 5$, then a set of 4 vectors in V cannot be linearly independent.

   (d) If V is infinite-dimensional, then any infinite linearly-independent subset is a basis.

   (e) $T : V \to W$ is a linear transformation, then $T(\mathbf{0}_V) = \mathbf{0}_W$.

   (f) If $T : V \to W$ has $T(\mathbf{a} + \mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b})$ for every $\mathbf{a}, \mathbf{b} \in V$ then T is a linear transformation.

   (g) If $T : V \to W$ has $T(r\mathbf{a}) = rT(\mathbf{a})$ for every $r \in F$ and every $\mathbf{a} \in V$ then T is a linear transformation.

   (h) For any $\mathbf{v}_1, \mathbf{v}_2 \in V$ and any $\mathbf{w}_1, \mathbf{w}_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.

   (i) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 2 and whose rank is 2.

   (j) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 4 and whose rank is 1.

   (k) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 1 and whose rank is 4.

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2. Find a basis for, and the dimension of, each of the following vector spaces:

   (a) The space of 3×3 symmetric matrices over $F = \mathbb{C}$.

   (b) The row space, column space, and nullspace of $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$ over $\mathbb{R}$.

   (c) The vectors in $\mathbb{Q}^5$ of the form $\langle a, b, c, d, e \rangle$ with $e = a + b$ and $b = c = d$, over $\mathbb{Q}$.

   (d) The row space, column space, and nullspace of $M = \begin{bmatrix} 1 & 3 & -2 & -6 & 8 \\ 2 & -1 & 2 & 8 & 1 \\ -1 & 1 & 1 & -3 & 3 \end{bmatrix}$ over $\mathbb{C}$.

   (e) The polynomials $p(x)$ in $P_4(\mathbb{R})$ such that $p(1) = 0$.

   (f) The matrices A in $M_{2\times 2}(\mathbb{Q})$ such that $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

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3. For each map $T : V \to W$, determine whether or not T is a linear transformation from V to W, and if it is not, identify at least one property that fails:

   (a) $V = W = \mathbb{R}^4$, $T(a, b, c, d) = (a - b, b - c, c - d, d - a)$.

   (b) $V = W = \mathbb{R}^2$, $T(a, b) = (a, b^2)$.

   (c) $V = W = M_{2\times 2}(\mathbb{Q})$, $T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} A - A \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.

   (d) $V = W = \mathbb{C}[x]$, $T(p(x)) = p(x^2) - xp'(x)$.

   (e) $V = W = M_{4\times 4}(\mathbb{F}_2)$, $T(A) = Q^{-1}AQ$, for a fixed 4×4 matrix Q.

   (f) $V = W = M_{4\times 4}(\mathbb{R})$, $T(A) = A^{-1}QA$, for a fixed 4×4 matrix Q.

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4. For each linear transformation $T : V \to W$, (i) find bases for the kernel and image of T, (ii) compute the nullity and rank of T and verify the conclusion of the nullity-rank theorem, (iii) identify whether T is one-to-one, and (iv) identify whether T is onto.

   (a) $T : \mathbb{Q}^2 \to \mathbb{Q}^3$ defined by $T(a, b) = \langle a + b, 2a + 2b, a + b\rangle$.

   (b) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$.

   (c) $T : P_2(\mathbb{C}) \to P_3(\mathbb{C})$ defined by $T(p) = xp(x) + p'(x)$.

   (d) $T : P_3(\mathbb{F}_3) \to P_4(\mathbb{F}_3)$ defined by $T(p) = x^3 p''(x)$. [Warning: Note that $3 = 0$ in $\mathbb{F}_3$.]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose $\dim(V) = n$ and that $T : V \to V$ is a linear transformation with $T^2 = 0$: in other words, that $T(T(\mathbf{v})) = \mathbf{0}$ for every vector $\mathbf{v} \in V$.

   (a) Show that $\text{im}(T)$ is a subspace of $\ker(T)$.

   (b) Show that $\dim(\text{im}(T)) \le n/2$.

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6. A linear transformation $T : V \to V$ such that $T^2 = T$ is called a projection map. The goal of this problem is to give some other descriptions of projection maps.

   (a) Suppose that $T : V \to V$ has the property that there exists a subspace W such that $\text{im}(T) = W$ and T is the identity map when restricted to W. Show that T is a projection map (it is called a projection onto the subspace W).

   (b) Conversely, suppose T is a projection map. Show that T is a projection onto the subspace $W = \text{im}(T)$.

   (c) Suppose that T is a projection map. Prove that $V = \ker(T) \oplus \text{im}(T)$. [Hint: Write $\mathbf{v} = [\mathbf{v} - T(\mathbf{v})] + T(\mathbf{v})$.]

   Remark: Projection maps are so named because they represent the geometric idea of projection. For example, in the event that $W = \text{im}(T)$ is one-dimensional, the corresponding projection map T represents projecting onto that line.

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7. Let F be a field and let V be the vector space of infinite sequences $\{a_n\}_{n\ge 1} = (a_1, a_2, a_3, a_4, \dots)$ of elements of F. Define the left-shift operator $L : V \to V$ via $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, a_5, \dots)$ and the right-shift operator $R : V \to V$ via $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

   (a) Show that L is a linear transformation that is onto but not one-to-one.

   (b) Show that R is a linear transformation that is one-to-one but not onto.

   (c) Deduce that on infinite-dimensional vector spaces, the conditions of being one-to-one and being onto are not in general equivalent.

   (d) Verify that $L \circ R$ is the identity map on V, but that $R \circ L$ is not the identity map on V.

   (e) Deduce that on infinite-dimensional vector spaces, a linear transformation with a left inverse or a right inverse need not have a two-sided inverse.

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8. [Challenge] The goal of this problem is to demonstrate some bizarre things one can do with infinite bases.

   (a) Show that $\dim_{\mathbb{Q}} \mathbb{R} = \dim_{\mathbb{Q}} \mathbb{C}$. Deduce that there exists a $\mathbb{Q}$-vector space isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$. [Hint: Use the fact that finite-dimensional $\mathbb{Q}$-vector spaces are countable.]

   We will now use this isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$ to define a different vector space structure on $\mathbb{C}$.

   (b) Let V be the set of complex numbers with the addition operation $z_1 \oplus z_2 = z_1 + z_2$ and scalar multiplication defined as follows: for $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$, set $\alpha \odot z = \varphi^{-1}[\alpha\varphi(z)]$. Show $(V, \oplus, \odot)$ is an $\mathbb{R}$-vector space.

   (c) Using the vector space structure defined in (b), show that $\dim_{\mathbb{R}} V = 1$.

   Remark: The point of (c) is that by changing the definition of scalar multiplication, we can make $\mathbb{C}$ into a 1-dimensional $\mathbb{R}$-vector space. By doing a similar thing in the reverse order, we could even make $\mathbb{R}$ into a 2-dimensional $\mathbb{C}$-vector space.
