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1. Identify each of the following statements as true or false:

   (a) The span of the empty set is the empty set.
   - $\boxed{\text{False}}$: the span of the empty set is the zero subspace containing only $\mathbf{0}$, which is different from the empty set.

   (b) The span of the zero vector is the zero subspace.
   - $\boxed{\text{True}}$: the only vector spanned by the zero vector is the zero vector itself.

   (c) If S is any subset of V, then $\text{span}(S)$ is the intersection of all subspaces of V containing S.
   - $\boxed{\text{True}}$: this is one of the basic facts about spans.

   (d) If S is any subset of V, then $\text{span}(S)$ always contains the zero vector.
   - $\boxed{\text{True}}$: the span is always a subspace, so it contains the zero vector (even if S is empty).

   (e) Any set containing the zero vector is linearly independent.
   - $\boxed{\text{False}}$: in fact the opposite is true, any set containing the zero vector is linearly dependent!

   (f) Any subset of a linearly independent set is linearly independent.
   - $\boxed{\text{True}}$: any linear dependence in the subset would give one in the original set.

   (g) Any subset of a linearly dependent set is linearly dependent.
   - $\boxed{\text{False}}$: removing elements from a linearly dependent set could certainly yield a linearly independent set (e.g., the empty set!).

   (h) The zero vector space has no basis.
   - $\boxed{\text{False}}$: the empty set is a basis for the zero space.

   (i) The set $\{\mathbf{0}\}$ is a basis for the zero vector space.
   - $\boxed{\text{False}}$: the empty set is a basis for the zero space.

   (j) Every vector space has a finite basis.
   - $\boxed{\text{False}}$: there are infinite-dimensional vector spaces, like $F[x]$, which have no finite basis.

   (k) Every vector space has a unique basis.
   - $\boxed{\text{False}}$: most vector spaces (e.g., F^2) have many different bases.

   (l) Every subspace of a finite-dimensional vector space is finite-dimensional.
   - $\boxed{\text{True}}$: if W is a subspace of V then $\dim(W) \le \dim(V)$, so $\dim(W)$ is also finite.

   (m) Every subspace of an infinite-dimensional vector space is infinite-dimensional.
   - $\boxed{\text{False}}$: infinite-dimensional spaces have many finite-dimensional subspaces (e.g., the zero subspace).

   (n) If $V = M_{m\times n}(F)$, then $\dim_F V = mn$.
   - $\boxed{\text{True}}$: a basis is given by the mn matrices with a 1 in one entry and 0s elsewhere.

   (o) If $V = F[x]$, then $\dim_F V$ is undefined.
   - $\boxed{\text{False}}$: $\dim_F F[x]$ is infinite but perfectly well-defined.

   (p) If $V = P_n(F)$, then $\dim_F V = n$.
   - $\boxed{\text{False}}$: $\dim_F V = n+1$, not n.

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2. For each set of vectors in each vector space, determine whether (i) they span V, (ii) they are linearly independent, and (iii) they are a basis:

   (a) $\langle 1, 2\rangle$ in $\mathbb{R}^2$.

   - Since $\mathbb{R}^2$ is 2-dimensional, this one vector $\boxed{\text{does not span}}$, and $\boxed{\text{is linearly independent}}$ but $\boxed{\text{not a basis}}$.

   (b) $\langle 1, 2\rangle$, $\langle 3, 2\rangle$, $\langle 1, 1\rangle$ in $\mathbb{R}^2$.

   - Some quick calculations will show that these vectors $\boxed{\text{do span}}$, but are $\boxed{\text{not linearly independent}}$ hence are $\boxed{\text{not a basis}}$.

   (c) $\langle 1, 2, 4\rangle$, $\langle 3, 2, 1\rangle$, $\langle 1, 1, 1\rangle$ in $\mathbb{R}^3$.

   - Since we have 3 vectors in $\mathbb{R}^3$ we can use the determinant shortcut: the matrix whose columns are the three given vectors has nonzero determinant, so the three vectors $\boxed{\text{do span}}$ and $\boxed{\text{are linearly independent}}$ hence $\boxed{\text{are a basis}}$.

   (d) $1 + x$, $x + x^2$ in $P_2(\mathbb{C})$.

   - These 2 polynomials $\boxed{\text{do not span}}$ because $P_2(\mathbb{C})$ has dimension 3. They $\boxed{\text{are linearly independent}}$, however, because neither is a scalar multiple of the other, but are $\boxed{\text{not a basis}}$.

   (e) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_5)$. [Note $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$; the entries of the matrices are considered modulo 5.]

   - These 2 matrices $\boxed{\text{do not span}}$ because $M_{2\times 2}$ has dimension 4. They $\boxed{\text{are linearly independent}}$, however, because neither is a scalar multiple of the other. They are $\boxed{\text{not a basis}}$.

   (f) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$.

   - These 3 matrices $\boxed{\text{do not span}}$ because $M_{2\times 2}$ has dimension 4. They $\boxed{\text{are linearly independent}}$, because $a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ yields $a + c = a + b = b + c = 0$ which has only the solution $a = b = c = 0$. They are $\boxed{\text{not a basis}}$.

   (g) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_2)$.

   - These 3 matrices $\boxed{\text{do not span}}$ because $M_{2\times 2}$ has dimension 4. They are $\boxed{\text{not linearly independent}}$ because their sum is the zero matrix. They are $\boxed{\text{not a basis}}$.

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3. Suppose V is a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$.

   (a) If S is linearly independent, show that T is linearly independent.

   - Suppose we had a dependence $a(\mathbf{v}_1) + b(\mathbf{v}_1 + \mathbf{v}_2) + c(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}$.
   - Distributing yields $(a + b + c)\mathbf{v}_1 + (b + c)\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$, and so linear independence of S requires $a+b+c = b+c = c = 0$, which has only the solution $a = b = c = 0$. Hence T is linearly independent.

   (b) If S spans V, show that T spans V.

   - If S spans V then for any $\mathbf{w} \in V$ we can write $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$.
   - Then it is not hard to see that $\mathbf{w} = (a - b)\mathbf{v}_1 + (b - c)(\mathbf{v}_1 + \mathbf{v}_2) + c(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$, so T also spans V.

   (c) If S is a basis of V, show that T is a basis of V.

   - Immediate from (a) and (b): if S is a basis then S is linearly independent and spans V, hence by (a) and (b) respectively T is linearly independent and spans V, so it is a basis.

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4. Suppose that $f_0, f_1, \dots, f_n$ are real-valued functions of x, all of which are n times differentiable. The Wronskian $W(f_0, f_1, \dots, f_n)$ is defined to be the determinant $W(f_0, f_1, \dots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$. For example, $W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$ and $W(x^2, 2x^2) = \begin{vmatrix} x^2 & 2x^2 \\ 2x & 4x \end{vmatrix} = 0$.

(a) Show that if $f_0, f_1, \dots, f_n$ are linearly dependent, then their Wronskian is zero.

- Suppose that $a_0 f_0 + a_1 f_1 + \cdots + a_n f_n = 0$. Then by taking derivatives we also have $a_0 f_0' + a_1 f_1' + \cdots + a_n f_n' = 0$ and similarly for the higher derivatives.
- This means $\begin{bmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{0}$. But as we have shown, if there is a nonzero vector $\mathbf{v}$ with $M\mathbf{v} = \mathbf{0}$, then $\det(M) = 0$: hence the Wronskian is zero as claimed.

(b) Deduce that if functions $f_0, f_1, \dots, f_n$ have a nonzero Wronskian, then they are linearly independent. Is the converse true? [Hint: Try $f_0 = x^2$ and $f_1 = x\,|x|$.]

- The first part is just the contrapositive of part (a).
- For the second part, observe that for $f_0 = x^2$ and $f_1 = x\,|x|$ we have $f_0' = 2x$ and $f_1' = 2\,|x|$ (this is straightforward to check with a graph or the definition of the derivative). Then $W(f_0, f_1) = \begin{vmatrix} x^2 & x\,|x| \\ 2x & 2\,|x| \end{vmatrix} = x^2 \cdot 2\,|x| - 2x \cdot x\,|x| = 0$.
- However, f_0 and f_1 are linearly independent: if $ax^2 + bx\,|x| = 0$ then setting $x = 1$ yields $a + b = 0$ and setting $x = -1$ yields $a - b = 0$, so that $a = b = 0$.

(c) Show that $\{1, \sin x, \cos x\}$ is a linearly independent set.

- We simply compute the Wronskian: it is $W(1, \sin x, \cos x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -\cos^2 x - \sin^2 x = -1$. Since this is nonzero, by (b) we conclude that the functions are linearly independent.

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5. Let V be a vector space such that $\dim_{\mathbb{C}} V = n$. Prove that if V is now considered a vector space over $\mathbb{R}$ (using the same addition and scalar multiplication), then $\dim_{\mathbb{R}} V = 2n$.

- Suppose that $\dim_{\mathbb{C}} V = n$. Choose a basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ for V, as a complex vector space. We claim that the set $S' = \{\mathbf{v}_1, i\mathbf{v}_1, \mathbf{v}_2, i\mathbf{v}_2, \dots, \mathbf{v}_n, i\mathbf{v}_n\}$ is a basis for V considered as a real vector space, which would immediately imply that $\dim_{\mathbb{R}} V = 2n$.
- S' spans V: Let $\mathbf{x}$ be a vector in V. By the assumption that S spans V, there exist (complex) scalars with $\mathbf{x} = (a_1 + b_1 i)\mathbf{v}_1 + \cdots + (a_n + b_n i)\mathbf{v}_n$. But by rearranging, we see that $\mathbf{x} = a_1\mathbf{v}_1 + b_1(i\mathbf{v}_1) + \cdots + a_n\mathbf{v}_n + b_n(i\mathbf{v}_n)$, meaning that $\mathbf{x}$ is a linear combination of the vectors in S' using real coefficients, as required.
- S' is linearly independent: Suppose there exist real numbers a_i, b_i such that $\mathbf{0} = a_1\mathbf{v}_1 + b_1(i\mathbf{v}_1) + \cdots + a_n\mathbf{v}_n + b_n(i\mathbf{v}_n)$. By rearranging this yields $\mathbf{0} = (a_1 + b_1 i)\mathbf{v}_1 + \cdots + (a_n + b_n i)\mathbf{v}_n$. But by the hypothesis that S is linearly independent, all of these (complex) coefficients must be zero.

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6. Let W be a vector space. Recall that if A and B are two subspaces of W then their sum is the set $A + B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}$.

   (a) Suppose that $A \cap B = \{\mathbf{0}\}$. If α is a basis for A and β is a basis for B, prove that α and β are disjoint and that $\alpha \cup \beta$ is a basis for $A + B$.

   - First, we show that $\alpha \cup \beta$ spans $A + B$: for any $\mathbf{a} + \mathbf{b} \in A + B$, since $\mathbf{a} \in A$ we can write $\mathbf{a} = a_1\mathbf{a}_1 + \cdots + a_k\mathbf{a}_k$ for some $\mathbf{a}_i \in \alpha$ and likewise since $\mathbf{b} \in B$ we can write $\mathbf{b} = b_1\mathbf{b}_1 + \cdots + b_l\mathbf{b}_l$ for some $\mathbf{b}_i \in \beta$.
   - Then $\mathbf{a} + \mathbf{b} = a_1\mathbf{a}_1 + \cdots + a_k\mathbf{a}_k + b_1\mathbf{b}_1 + \cdots + b_l\mathbf{b}_l$ is in $\text{span}(\alpha \cup \beta)$, as required.
   - Now suppose we had a dependence $a_1\mathbf{a}_1 + \cdots + a_k\mathbf{a}_k + b_1\mathbf{b}_1 + \cdots + b_l\mathbf{b}_l = \mathbf{0}$.
   - Then rearranging gives $a_1\mathbf{a}_1 + \cdots + a_k\mathbf{a}_k = -(b_1\mathbf{b}_1 + \cdots + b_l\mathbf{b}_l)$. Since the left-hand side is a vector in A and the right-hand side is a vector in B, the common expression is an element in $A \cap B$.
   - But then since $A \cap B = \{\mathbf{0}\}$, we have $a_1\mathbf{a}_1 + \cdots + a_k\mathbf{a}_k = \mathbf{0} = -(b_1\mathbf{b}_1 + \cdots + b_l\mathbf{b}_l)$.
   - Then since α and β are both linearly independent, we have $a_1 = \cdots = a_k = 0$ and $b_1 = \cdots = b_l = 0$. Hence $\alpha \cup \beta$ is linearly independent, as claimed.

   (b) Now suppose that α is a basis for A and β is a basis for B. If $\alpha \cup \beta$ is a basis for $A + B$ and α and β are disjoint, prove that $A \cap B = \{\mathbf{0}\}$.

   - Suppose that $\mathbf{v} \in A \cap B$. Then since $\mathbf{v} \in A$ we may write $\mathbf{v} = a_1\mathbf{a}_1 + \cdots + a_k\mathbf{a}_k$ for some $\mathbf{a}_i \in \alpha$, and since $\mathbf{v} \in B$ we may also write $\mathbf{v} = b_1\mathbf{b}_1 + \cdots + b_l\mathbf{b}_l$ for some $\mathbf{b}_i \in \beta$.
   - Then subtracting the expressions yields $a_1\mathbf{a}_1 + \cdots + a_k\mathbf{a}_k - b_1\mathbf{b}_1 - \cdots - b_l\mathbf{b}_l = \mathbf{0}$.
   - But since $\alpha \cup \beta$ is a basis and α, β are linearly independent, all of the scalar coefficients must be zero. Then $\mathbf{v} = 0\mathbf{a}_1 + \cdots + 0\mathbf{a}_k = \mathbf{0}$, and so $A \cap B = \{\mathbf{0}\}$.

The situation in (a)-(b) is very important and arises often. Explicitly, if A and B are two subspaces of W such that $A + B = W$ and $A \cap B = \{\mathbf{0}\}$ is the trivial subspace, we write $W = A \oplus B$ and call W the (internal) direct sum of A and B. (The idea is that we may "decompose" W into two independent pieces A and B.)

   (c) Show that $\mathbb{R}^2$ is the direct sum of the subspaces given by the x-axis and the y-axis, and is also the direct sum of the subspaces given by the x-axis and the line $y = 3x$.

   - First for $X = \{\langle x, 0\rangle\}$ and $Y = \{\langle 0, y\rangle\}$, since $\langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle$, we have $\mathbb{R}^2 = X + Y$. Furthermore, $X \cap Y = \{\langle 0, 0\rangle\}$ so the sum is a direct sum.
   - Second with $Z = \{\langle y/3, y\rangle\}$, since $\langle x, y\rangle = \langle x - y/3, 0\rangle + \langle y/3, y\rangle$, we have $\mathbb{R}^2 = X + Z$. Furthermore, $X \cap Z = \{\langle 0, 0\rangle\}$ so the sum is a direct sum.

   (d) Prove that $W = A \oplus B$ if and only if every vector $\mathbf{w} \in W$ can be written uniquely in the form $\mathbf{w} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} \in A$ and $\mathbf{b} \in B$.

   - First suppose that every vector $\mathbf{w} \in W$ can be written uniquely in the form $\mathbf{w} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} \in A$ and $\mathbf{b} \in B$. Then clearly $W = A + B$.
   - Furthermore, for any $\mathbf{w} \in A \cap B$, we may write $\mathbf{w} = \mathbf{w} + \mathbf{0} = \mathbf{0} + \mathbf{w}$. Hence by uniqueness, we must have $\mathbf{w} = \mathbf{0}$, and therefore $A \cap B = \{\mathbf{0}\}$. Hence $W = A \oplus B$ as required.
   - Conversely, suppose $W = A \oplus B$. Then every vector $\mathbf{w} \in W$ can be written as $\mathbf{a} + \mathbf{b}$, so we need only consider uniqueness: if $\mathbf{w} = \mathbf{a} + \mathbf{b} = \mathbf{a}' + \mathbf{b}'$, then subtracting yields $\mathbf{a} - \mathbf{a}' = \mathbf{b}' - \mathbf{b}$.
   - Since the vector $\mathbf{a} - \mathbf{a}' = \mathbf{b}' - \mathbf{b}$ is in both A and B, it must be the zero vector: hence $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$, so the representation of $\mathbf{w}$ is unique as required.

   (e) Show that if $W = A \oplus B$ then $\dim(W) = \dim(A) + \dim(B)$. Show using an explicit counterexample that the converse statement need not hold.

   - The first part is simply a rewriting of part (b). For the second part, if for example we take $W = \mathbb{R}^2$ with $A = B = \text{span}(\langle 1, 0\rangle)$, then the dimensions sum correctly, but $A + B = A = \text{span}(\langle 1, 0\rangle) \neq W$.

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7. Let F be a finite field with q elements. The goal of this problem is to count the invertible matrices in $M_{n\times n}(F)$.

   (a) Suppose W is a k-dimensional subspace of F^n. Show that W contains exactly q^k vectors.

   - Choose a basis $\mathbf{w}_1, \dots, \mathbf{w}_k$ for W. Then every vector in W may be written as a linear combination $a_1\mathbf{w}_1 + \cdots + a_k\mathbf{w}_k$ for unique scalars $a_1, \dots, a_k$.
   - Since there are q choices for each scalar a_i for each $1 \le i \le k$, there are q^k total choices for the coefficients and thus q^k possible vectors.

   (b) Show that the number of invertible $n \times n$ matrices in $M_{n\times n}(F)$ is equal to the number of ordered lists $\mathbf{v}_1$, $\mathbf{v}_2$, ... , $\mathbf{v}_n$ of n linearly independent vectors from F^n.

   - From our discussion of bases of F^n, we know that a list $\mathbf{v}_1$, $\mathbf{v}_2$, ... , $\mathbf{v}_n$ of n vectors in F^n is linearly independent if and only if it is a basis.
   - Also from our discussion, we know that the matrix whose columns are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... , $\mathbf{v}_n$ is invertible if and only if the columns are linearly independent.
   - Thus, by combining these two facts, we see that the number of invertible $n \times n$ matrices in $M_{n\times n}(F)$ is equal to the number of ordered lists $\mathbf{v}_1$, $\mathbf{v}_2$, ... , $\mathbf{v}_n$ of n linearly independent vectors from F^n.

   (c) For any integer $0 \le k \le n$, show that there are exactly $(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})$ ordered lists $\mathbf{v}_1$, $\mathbf{v}_2$, ... , $\mathbf{v}_k$ of k linearly independent vectors from F^n. [Hint: Count the number of ways to choose the vector $\mathbf{v}_i$ not in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1})$ for each i.]

   - Since subsets of linearly independent sets are linearly independent, we see that each of $\{\mathbf{v}_1\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$, ... , $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is linearly independent.
   - So we simply need to count the number of selections of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... , $\mathbf{v}_k$ (in order) where each of the sets $\{\mathbf{v}_1\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$, ... , $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is linearly independent.
   - If we have chosen $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$, then $\mathbf{v}_i$ may be any vector not in $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1})$. By part (a), there are q^{i-1} vectors in this span, so there are $q^n - q^{i-1}$ possible choices for $\mathbf{v}_i$.
   - Multiplying, we obtain the total number of lists as $(q^n - 1)(q^n - q)\cdots(q^n - q^k)$, as required.

   (d) Deduce that the number of invertible $n \times n$ matrices in $M_{n\times n}(F)$ is equal to $(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})\cdots(q^n - q^{n-1})$. In particular, find the number of invertible 5×5 matrices over the field $\mathbb{F}_2$.

   - The first result follows immediately from setting $k = n$ in part (c).
   - For the second part, taking $n = 5$ and $q = 2$ yields the total $31 \cdot 30 \cdot 28 \cdot 24 \cdot 16 = 9999360$.

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8. [Challenge] Zorn's lemma states that if $\mathcal{F}$ is a nonempty partially-ordered set in which every chain (a subset with $A \le B$ or $B \le A$ for all A, B in the subset) has an upper bound (an element $U \in \mathcal{F}$ such that $X \le U$ for all X in the chain), then $\mathcal{F}$ contains a maximal element (an element $M \in \mathcal{F}$ such that if $M \le Y$ for some $Y \in \mathcal{F}$, then in fact $Y = M$). The goal of this problem is to use Zorn's lemma to prove that any linearly independent set can be extended to a basis and that any spanning set contains a basis.

   (a) Suppose that S is a maximal linearly-independent subset of a vector space V (this means that if T is any linearly-independent subset of V containing S, then in fact $T = S$). Prove that S is a basis of V.

   - Suppose S does not span V: then there exists a vector $\mathbf{w} \in V$ such that $\mathbf{w} \notin \text{span}(S)$.
   - Then by our results on linear independence and span, the set $S \cup \{\mathbf{w}\}$ would also be linearly independent. But this contradicts the assumption that S is maximal, since $T = S \cup \{\mathbf{w}\}$ would be a linearly independent set containing S with $T \ne S$.
   - This is impossible, so there cannot exist any vector $\mathbf{w} \in V$ with $\mathbf{w} \notin \text{span}(S)$. Hence S spans V, so since it is linearly independent, it is a basis.

   (b) Suppose $\mathcal{C}$ is a chain of linearly independent subsets of V (i.e., a collection of linearly independent subsets with the property that $A \subseteq B$ or $B \subseteq A$ for any $A, B \in \mathcal{C}$). Show that $U = \bigcup_{A\in\mathcal{C}} A$ is also linearly independent. [Hint: A linear dependence can only involve finitely many vectors.]

   - Suppose that there exist vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in U and scalars $a_1, \dots, a_k$ such that $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$.
   - Suppose the vectors $\mathbf{v}_i$ lie in the subsets $A_i \in \mathcal{C}$. Then since $\mathcal{C}$ is a chain, since $A_i \subseteq A_j$ for each pair (i, j), by a trivial induction we see that one of the A_i must contain all of the others, hence contains all of the vectors $\mathbf{v}_i$.

- But since this subset A_i is linearly independent, we must have $a_1 = \cdots = a_k = 0$, and so $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent.
- Thus, since every finite subset of U is linearly independent, U is linearly independent.

(c) Prove that every linearly independent subset of V can be extended to a basis.

- Let $\mathcal{F}$ be the collection of all linearly-independent subsets of V containing the given set, partially ordered by inclusion. Note that $\mathcal{F}$ is not empty since it contains the original linearly independent set.
- By part (b), if $\mathcal{C}$ is any chain in $\mathcal{F}$ contains a maximal element. Such a maximal element is a maximal linearly-independent subset of V, which by part (a) is a basis of V.

(d) Suppose that S is a minimal spanning set of a vector space V (this means that if T is any subset of S that spans V, then in fact $T = S$). Prove that S is a basis of V.

- Suppose S is not a basis of V: then necessarily S must be linearly dependent, so there exists vector $\mathbf{w} \in S$ such that $\mathbf{w} \in \operatorname{span}(S')$ where $S' = S \backslash \{\mathbf{w}\}$.
- Then by our results on span, we have $\operatorname{span}(S') = \operatorname{span}(S) = V$, but S' is a proper subset of S, contradicting minimality. Hence S must be linearly independent hence a basis.

(e) Let $V = \mathbb{Q}$ with scalar field $F = \mathbb{Q}$ and let $S_n = \{n, n+1, n+2, \dots\}$ for each positive integer n. Show that each set S_n is a spanning set and that the sets S_n form a chain, but that the intersection $\bigcap_{n=1}^{\infty} S_n$ is not a spanning set.

- Since V is 1-dimensional, any set containing something other than 0 automatically spans V. Since each S_n is infinite, they all span V, and clearly $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ so they form a chain.
- However, the intersection $\bigcap_{n=1}^{\infty} S_n$ is empty, because any positive integer k is not in S_{k+1}, and so the intersection is not a spanning set.

Remark: It is natural to try to use Zorn's lemma to prove that a minimal spanning set must exist, in analogy to (b). The obvious approach for this does not work, as (e) shows: the intersection of a chain of spanning sets need not span V! (A valid way to show every spanning set contains a basis is to apply Zorn's lemma to the linearly independent subsets of the spanning set.)
