Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:

   (a) The span of the empty set is the empty set.
   (b) The span of the zero vector is the zero subspace.
   (c) If S is any subset of V, then $\text{span}(S)$ is the intersection of all subspaces of V containing S.
   (d) If S is any subset of V, then $\text{span}(S)$ always contains the zero vector.
   (e) Any set containing the zero vector is linearly independent.
   (f) Any subset of a linearly independent set is linearly independent.
   (g) Any subset of a linearly dependent set is linearly dependent.
   (h) The zero vector space has no basis.
   (i) The set $\{\mathbf{0}\}$ is a basis for the zero vector space.
   (j) Every vector space has a finite basis.
   (k) Every vector space has a unique basis.
   (l) Every subspace of a finite-dimensional vector space is finite-dimensional.
   (m) Every subspace of an infinite-dimensional vector space is infinite-dimensional.
   (n) If $V = M_{m\times n}(F)$, then $\dim_F V = mn$.
   (o) If $V = F[x]$, then $\dim_F V$ is undefined.
   (p) If $V = P_n(F)$, then $\dim_F V = n$.

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2. For each set of vectors in each vector space, determine whether (i) they span V, (ii) they are linearly independent, and (iii) they are a basis:

   (a) $\langle 1,2\rangle$ in $\mathbb{R}^2$.
   (b) $\langle 1,2\rangle$, $\langle 3,2\rangle$, $\langle 1,1\rangle$ in $\mathbb{R}^2$.
   (c) $\langle 1,2,4\rangle$, $\langle 3,2,1\rangle$, $\langle 1,1,1\rangle$ in $\mathbb{R}^3$.
   (d) $1+x$, $x+x^2$ in $P_2(\mathbb{C})$.
   (e) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_5)$. [Note $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$; the entries are considered modulo 5.]
   (f) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$.
   (g) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_2)$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Suppose V is a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$.

   (a) If S is linearly independent, show that T is linearly independent.
   (b) If S spans V, show that T spans V.
   (c) If S is a basis of V, show that T is a basis of V.

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4. Suppose that $f_0, f_1, \ldots, f_n$ are real-valued functions of x, all of which are n times differentiable. The Wronskian $W(f_0, f_1, \ldots, f_n)$ is defined to be the determinant $W(f_0, f_1, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$. For example, $W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$ and $W(x^2, 2x^2) = \begin{vmatrix} x^2 & 2x^2 \\ 2x & 4x \end{vmatrix} = 0$.

(a) Show that if $f_0, f_1, \ldots, f_n$ are linearly dependent, then their Wronskian is zero.

(b) Deduce that if functions $f_0, f_1, \ldots, f_n$ have a nonzero Wronskian, then they are linearly independent. Is the converse true? [Hint: Try $f_0 = x^2$ and $f_1 = x\,|x|$.]

(c) Show that $\{1, \sin x, \cos x\}$ is a linearly independent set.

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5. Let V be a vector space such that $\dim_{\mathbb{C}} V = n$. Prove that if V is now considered a vector space over $\mathbb{R}$ (using the same addition and scalar multiplication), then $\dim_{\mathbb{R}} V = 2n$.

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6. Let W be a vector space. Recall that if A and B are two subspaces of W then their sum is the set $A + B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}$.

(a) Suppose that $A \cap B = \{\mathbf{0}\}$. If α is a basis for A and β is a basis for B, prove that α and β are disjoint and that $\alpha \cup \beta$ is a basis for $A + B$.

(b) Now suppose that α is a basis for A and β is a basis for B. If $\alpha \cup \beta$ is a basis for $A + B$ and α and β are disjoint, prove that $A \cap B = \{\mathbf{0}\}$.

The situation in (a)-(b) is very important and arises often. Explicitly, if A and B are two subspaces of W such that $A + B = W$ and $A \cap B = \{\mathbf{0}\}$ is the trivial subspace, we write $W = A \oplus B$ and call W the (internal) direct sum of A and B. (The idea is that we may "decompose" W into two independent pieces A and B.)

(c) Show that $\mathbb{R}^2$ is the direct sum of the subspaces given by the x-axis and the y-axis, and is also the direct sum of the subspaces given by the x-axis and the line $y = 3x$.

(d) Prove that $W = A \oplus B$ if and only if every vector $\mathbf{w} \in W$ can be written uniquely in the form $\mathbf{w} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} \in A$ and $\mathbf{b} \in B$.

(e) Show that if $W = A \oplus B$ then $\dim(W) = \dim(A) + \dim(B)$. Show using an explicit counterexample that the converse statement need not hold.

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7. Let F be a finite field with q elements. The goal of this problem is to count the invertible matrices in $M_{n \times n}(F)$.

(a) Suppose W is a k-dimensional subspace of F^n. Show that W contains exactly q^k vectors.

(b) Show that the number of invertible $n \times n$ matrices in $M_{n \times n}(F)$ is equal to the number of ordered lists $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of n linearly independent vectors from F^n.

(c) For any integer $0 \le k \le n$, show that there are exactly $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$ ordered lists $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of k linearly independent vectors from F^n. [Hint: Count the number of ways to choose the vector $\mathbf{v}_i$ not in $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1})$ for each i.]

(d) Deduce that the number of invertible $n \times n$ matrices in $M_{n \times n}(F)$ is equal to $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}) \cdots (q^n - q^{n-1})$. In particular, find the number of invertible 5×5 matrices over the field $\mathbb{F}_2$.

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8. [Challenge] Zorn's lemma states that if $\mathcal{F}$ is a nonempty partially-ordered set in which every chain (a subset with $A \leq B$ or $B \leq A$ for all A, B in the subset) has an upper bound (an element $U \in \mathcal{F}$ such that $X \leq U$ for all X in the chain), then $\mathcal{F}$ contains a maximal element (an element $M \in \mathcal{F}$ such that if $M \leq Y$ for some $Y \in \mathcal{F}$, then in fact $Y = M$). The goal of this problem is to use Zorn's lemma to prove that any linearly independent set can be extended to a basis and that any spanning set contains a basis.

   (a) Suppose that S is a maximal linearly-independent subset of a vector space V (this means that if T is any linearly-independent subset of V containing S, then in fact $T = S$). Prove that S is a basis of V.

   (b) Suppose $\mathcal{C}$ is a chain of linearly independent subsets of V (i.e., a collection of linearly independent subsets with the property that $A \subseteq B$ or $B \subseteq A$ for any $A, B \in \mathcal{C}$). Show that $U = \bigcup_{A \in \mathcal{C}} A$ is also linearly independent. [Hint: A linear dependence can only involve finitely many vectors.]

   (c) Prove that every linearly independent subset of V can be extended to a basis.

   (d) Suppose that S is a minimal spanning set of a vector space V (this means that if T is any subset of S that spans V, then in fact $T = S$). Prove that S is a basis of V.

   (e) Let $V = \mathbb{Q}$ with scalar field $F = \mathbb{Q}$ and let $S_n = \{n, n+1, n+2, \dots\}$ for each positive integer n. Show that each set S_n is a spanning set and that the sets S_n form a chain, but that the intersection $\bigcap_{n=1}^{\infty} S_n$ is not a spanning set.

Remark: It is natural to try to use Zorn's lemma to prove that a minimal spanning set must exist, in analogy to (b). The obvious approach for this does not work, as (e) shows: the intersection of a chain of spanning sets need not span V! (A valid way to show every spanning set contains a basis is to apply Zorn's lemma to the linearly independent subsets of the spanning set.)
