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1. Identify each of the following statements as true or false:

   (a) There is a system of linear equations over $\mathbb{R}$ having exactly two different solutions.

   - $\boxed{\text{False}}$. Over $\mathbb{R}$ the only possibilities are that there are no solutions, exactly 1 solution, or infinitely many solutions.

   (b) If A is $n \times n$ and $\mathbf{c}$ is $n \times 1$, then the matrix system $A\mathbf{x} = \mathbf{c}$ always has a unique solution for $\mathbf{x}$.

   - $\boxed{\text{False}}$. There is only a unique solution when A is invertible.

   (c) Every vector space contains a zero vector.

   - $\boxed{\text{True}}$: this is one of the vector space axioms.

   (d) In any vector space, $\alpha\mathbf{v} = \alpha\mathbf{w}$ implies that $\mathbf{v} = \mathbf{w}$.

   - $\boxed{\text{False}}$: for example we have $0 \cdot \mathbf{v} = 0 \cdot \mathbf{w}$ for any $\mathbf{v}, \mathbf{w}$.

   (e) In any vector space, $\alpha\mathbf{v} = \beta\mathbf{v}$ implies that $\alpha = \beta$.

   - $\boxed{\text{False}}$: for example we have $1 \cdot \mathbf{0} = 2 \cdot \mathbf{0}$.

   (f) If U is a subspace of V and V is a subspace of W, then U is a subspace of W.

   - $\boxed{\text{True}}$: this follows from the subspace criterion (or even just the definition of subspace).

   (g) The empty set is a subspace of any vector space.

   - $\boxed{\text{False}}$: subspaces are by definition not empty.

   (h) The intersection of two subspaces is always a subspace.

   - $\boxed{\text{True}}$: the intersection of any collection of subspaces is a subspace.

   (i) The union of two subspaces is always a subspace.

   - $\boxed{\text{False}}$: for example, the union of $\{\langle x, 0\rangle\}$ and $\{\langle 0, y\rangle\}$ in $\mathbb{R}^2$ is not a subspace.

   (j) The union of two subspaces is never a subspace.

   - $\boxed{\text{False}}$: for example if we take two subspaces $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$ is still a subspace.

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2. Determine whether or not each given set S is a subspace of the given vector space V. For each set that is not a subspace, identify at least one part of the subspace criterion that fails.

   (a) $V = \mathbb{R}^4$, $S =$ the vectors $\langle a, b, c, d\rangle$ in $\mathbb{R}^4$ with $2a + 0b + 2c + 6d = 2026$.

   - This set $\boxed{\text{is not a subspace}}$ because it does not contain the zero vector. (It also fails the other two parts of the subspace criterion.)

   (b) $V = \mathbb{R}^4$, $S =$ the vectors $\langle a, b, c, d\rangle$ in $\mathbb{R}^4$ with $abcd = 0$.

   - This set $\boxed{\text{is not a subspace}}$ because it is not closed under addition, as for instance S contains $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ but not their sum $(1, 1, 1, 1)$.

   (c) $V = \mathbb{R}^4$, $S =$ the vectors $\langle a, b, c, d\rangle$ with $d = a + b + c$ and $a = c$.

   - This set $\boxed{\text{is a subspace}}$ because it contains the zero vector and is closed under addition and scalar multiplication.

   (d) $V =$ differentiable functions on $[0, 1]$, $S =$ the functions with $f'(x) = f(x)$.

   - This set $\boxed{\text{is a subspace}}$ because it contains the zero function and is closed under addition and scalar multiplication.

(e) V = real-valued functions on $\mathbb{R}$, S = the functions with $f(x) = f(1-x)$ for all real x.

- This set $\boxed{\text{is a subspace}}$ because it contains the zero function and is closed under addition and scalar multiplication.

(f) $V = M_{3\times 3}(\mathbb{R})$, S = the 3×3 matrices with integer entries.

- This set $\boxed{\text{is not a subspace}}$ because it is not closed under scalar multiplication (specifically, by non-integer scalars).

(g) $V = M_{3\times 3}(\mathbb{R})$, S = the 3×3 matrices with nonnegative real entries.

- This set $\boxed{\text{is not a subspace}}$ because it is not closed under scalar multiplication (specifically, by negative scalars).

(h) $V = P_5(\mathbb{R})$, S = the polynomials in V divisible by x^2.

- This set $\boxed{\text{is a subspace}}$ because it contains the zero polynomial and is closed under addition and scalar multiplication.

(i) $V = P_3(\mathbb{C})$, S = the polynomials in V with $p(i) = 0$.

- This set $\boxed{\text{is a subspace}}$ because it contains the zero polynomial and is closed under addition and scalar multiplication.

(j) $V = M_{2\times 2}(\mathbb{Q})$, S = the matrices in V of determinant zero.

- This set $\boxed{\text{is not a subspace}}$ because it is not closed under addition. For example, S contains $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ but not their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (It does satisfy the other two properties, however.)

(k) V = real-valued functions on $\mathbb{R}$, S = the functions that are zero at every rational number.

- This set $\boxed{\text{is a subspace}}$ because it contains the zero function and is closed under addition and scalar multiplication. (Note that V contains lots of functions, such as the function that is 1 at $x = \sqrt{2}$ and 0 everywhere else.)

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3. Show the following things, by induction or otherwise:

(a) Prove that $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = (n-1)n(n+1)/3$ for all integers $n \geq 2$.

- Induction on n. The base case $n = 2$ follows as $1 \cdot 2 = 1 \cdot 2 \cdot 3/3$.
- For the inductive step, suppose $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = (n-1)n(n+1)/3$. Then $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n + n(n+1) = (n-1)n(n+1)/3 + n(n+1) = n(n+1)[(n-1)/3 + 1] = n(n+1)(n+2)/3$, as required.

(b) Prove that the nth power of the matrix $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix}$ for each positive integer n.

- Induction on n. The base case $n = 1$ follows as $\begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ for $n = 1$.
- For the inductive step, suppose $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}^n = \begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix}$. Then $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}^{n+1}$ $= \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}^n = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix} = \begin{bmatrix} 1-2(n+1) & 4(n+1) \\ -(n+1) & 1+2(n+1) \end{bmatrix}$.

(c) Prove that a matrix with m rows can be put into row-echelon form using at most $m(m-1)/2$ elementary row operations. [Hint: If a matrix has $m+1$ rows, explain why it takes at most m row operations to clear out the entries in the first column.]

- Induction on m. The base case $m = 1$ is trivial because any matrix with one row is already in row-echelon form.
- For the inductive step, suppose that any matrix with m rows can be row-reduced using at most $m(m-1)/2$ row operations, and let A have $m+1$ rows. If A is the zero matrix we are already done. Otherwise, ignoring any leading columns of all zeroes, assume the first column has a nonzero entry.
- If the top left entry is nonzero, we need at most m subtractions of a multiple of the first row from the other rows to clear out the first column. Otherwise, if the top left entry is zero, we can use one row swap to make it nonzero, and then we need at most $m-1$ subtractions in order to clear out the entries in the first column (since at least one entry is zero, we don't need a subtraction in its row). Either way, we need at most m row operations.
- At this point, we need only row-reduce the $(m-1) \times (n-1)$ submatrix in the bottom right, which by hypothesis takes at most $m(m-1)/2$ row operations. So in total, we require at most $m + m(m-1)/2 = m(m+1)/2$ row operations, as required.

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4. The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Thus, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and so forth. By convention we also set $F_0 = 0$, satisfying the same recurrence.

(a) Prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ for every positive integer n.

- Induction on n. For the base case $n = 1$, we have $F_1 = 1 = F_2$ which is true.
- For the inductive step suppose that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$. Then $F_1 + F_3 + F_5 + \cdots + F_{2n-1} + F_{2n+1} = [F_1 + F_3 + F_5 + \cdots + F_{2n-1}] + F_{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2}$ as required.

(b) Prove that the nth power of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ for each positive integer n.

- Induction on n. The base case $n = 1$ follows as $\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
- For the inductive step, suppose $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n =$ $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n & F_n + F_{n-1} \\ F_{n+1} & F_n \end{bmatrix} = \begin{bmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix}$ as required.

(c) Prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for each positive integer n. [Hint: Use (b).]

- From determinant properties we have $\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = (-1)^n$ but by (b) since $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ this immediately yields $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, as claimed.

(d) Let M_n be the $n \times n$ matrix with 1s on the diagonal and directly below the diagonal, -1s directly above the diagonal, and 0s elsewhere. Prove that $\det(M_n)$ is the $(n+1)$st Fibonacci number F_{n+1}.

- Strong induction on n. The base cases $n = 1$ and $n = 2$ follow by observing that $M_1 = [1]$ so $\det(M_1) = 1 = F_2$ and that $M_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ so $\det(M_2) = 2 = F_3$, as required.
- For the inductive step, suppose $\det(M_{n-1}) = F_n$ and $\det(M_{n-2}) = F_{n-1}$ and consider expanding $\det(M_n)$ along the first row. Only the terms from the first and second entries contribute, since all other entries in the first row are zero. Deleting the first row and column of M_n yields M_{n-1}, while deleting the first row and second column of M_n yields a matrix whose first column has a 1 and then all zeroes, so its determinant is the same as the determinant obtained by deleting its first row and column, which results in the matrix M_{n-2}.
- Thus via expansion by minors we see $\det(M_n) = 1 \cdot \det(M_{n-1}) - (-1) \cdot \det(M_{n-2}) = F_n + F_{n-1} = F_{n+1}$.

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5. Suppose A is an $m \times n$ matrix with entries from the field F.

   (a) Show that the set of all vectors $\mathbf{x} \in F^n$ such that $A\mathbf{x}$ equals the zero vector (in F^m) is a subspace of F^n.

   - We simply check the subspace criterion:
   - For [S1], clearly $A\mathbf{0} = \mathbf{0}$.
   - For [S2], if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$ then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
   - For [S3], if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$ then $A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha\mathbf{0} = \mathbf{0}$.

   (b) Deduce that the set of solutions to any homogeneous system of linear equations (i.e., in which all of the constants are equal to zero) over F is an F-vector space.

   - If we take A to be the coefficient matrix, then the variable vector $\mathbf{x}$ is a simultaneous solution to all of the equations if and only if $A\mathbf{x} = \mathbf{0}$.
   - So by part (a), the space of solutions is a subspace of F^m hence is a vector space.

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6. If V is a vector space and W_1, W_2 are two subspaces of V, their sum is defined to be the set $W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 : \mathbf{w}_1 \in W_1 \text{ and } \mathbf{w}_2 \in W_2\}$ of all sums of an element of W_1 with an element of W_2.

   (a) Prove that $W_1 + W_2$ contains W_1 and W_2, and is a subspace of V.

   - For the first part, for any $\mathbf{w}_1$ in W_1 and $\mathbf{w}_2$ in W_2 we can write $\mathbf{w}_1 = \mathbf{w}_1 + \mathbf{0}$ and $\mathbf{w}_2 = \mathbf{0} + \mathbf{w}_2$. Thus so $\mathbf{w}_1$ and $\mathbf{w}_2$ are both in $W_1 + W_2$, and so $W_1 + W_2$ contains W_1 and W_2.
   - For the other part, we check the subspace criterion.
   - For [S1], $\mathbf{0} = \mathbf{0} + \mathbf{0}$ so $W_1 + W_2$ contains $\mathbf{0}$.
   - For [S2], suppose $\mathbf{a}_1 + \mathbf{b}_1$ and $\mathbf{a}_2 + \mathbf{b}_2$ are in $W_1 + W_2$. Then $\mathbf{a}_1 + \mathbf{a}_2$ is in W_1 (by the subspace criterion in W_1) and $\mathbf{b}_1 + \mathbf{b}_2$ is in W_2 (by the subspace criterion in W_2). So since $(\mathbf{a}_1 + \mathbf{b}_1) + (\mathbf{a}_2 + \mathbf{b}_2) = (\mathbf{a}_1 + \mathbf{a}_2) + (\mathbf{b}_1 + \mathbf{b}_2)$ we conclude that $(\mathbf{a}_1 + \mathbf{b}_1) + (\mathbf{a}_2 + \mathbf{b}_2)$ is in $W_1 + W_2$.
   - For [S3], suppose $\mathbf{a} + \mathbf{b}$ is in $W_1 + W_2$. Then $c\mathbf{a}$ is in W_1 and $c\mathbf{b}$ is in W_2 so $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ is in $W_1 + W_2$.

   (b) Show that if W is a subspace of V containing W_1 and W_2, then W must contain $W_1 + W_2$. Deduce that $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2.

   - Suppose W is a subspace of V containing both W_1 and W_2 and let $\mathbf{a} + \mathbf{b}$ be any vector in $W_1 + W_2$.
   - Since $\mathbf{a}$ is in W_1 and $\mathbf{b}$ is in W_2, both $\mathbf{a}$ and $\mathbf{b}$ are in W. So by the subspace criterion in W, $\mathbf{a} + \mathbf{b}$ is in W.
   - Since $\mathbf{a} + \mathbf{b}$ was an arbitrary element of $W_1 + W_2$, we conclude that $W_1 + W_2$ is contained in W.
   - Therefore, every subspace containing W_1 and W_2 contains $W_1 + W_2$. Since $W_1 + W_2$ is itself a subspace by (a), it is the smallest.

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7. [Challenge] Let F be a field and suppose $x_1, \ldots, x_n$ are elements of F. The goal of this problem is to evaluate the famous <u>Vandermonde determinant</u> $V(x_1, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$.

   (a) Show that if any of the x_i are equal to one another, then $V(x_1, \ldots, x_n) = 0$.

   - If $x_i = x_j$ then the ith and jth rows of the matrix are equal, so its determinant is zero.

   (b) Show that as a polynomial in the variables $x_1, \ldots, x_n$, $V(x_1, \ldots, x_n)$ has degree $\frac{n(n-1)}{2}$ and is divisible by $x_j - x_i$ for any $i \neq j$. [Hint: Use (a) and the remainder theorem.]

   - For the degree, observe that when we expand the determinant by minors, each term will be a product of one term from the first column, one from the second column, ... , and one from the last column, so the resulting product will have degree $0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$.
   - Now dividing $V(x_1, \ldots, x_n)$ by $x_i - x_j$ (where we think of x_i as the variable) leaves some remainder term. When we set $x_i = x_j$ then the remainder term vanishes by part (a), so the remainder must be the zero polynomial.

   (c) Deduce that $V(x_1, \ldots, x_n)$ is divisible by the product $\prod_{1 \leq i < j \leq n}(x_j - x_i)$ and that this product is a polynomial of degree $\frac{n(n-1)}{2}$.

   - By (b) applied to all possible pairs (i, j) with $1 \leq i < j \leq n$ we see that $x_j - x_i$ divides $V(x_1, \ldots, x_n)$. Since these terms are all relatively prime, their product must divide $V(x_1, \ldots, x_n)$.
   - Furthermore, the number of possible pairs (i, j) is simply $\binom{n}{2} = \frac{n(n-1)}{2}$ since we may pick any unordered pair of values $\{x_i, x_j\}$, so the product of these terms has degree $\frac{n(n-1)}{2}$.

   (d) Show in fact that $V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$. [Hint: Compare degrees and coefficients of $x_1^0 x_2^1 \cdots x_n^{n-1}$ on both sides.]

   - By (b) and (c) we see that dividing V by the product yields a polynomial of degree 0 (in other words, a constant). But since the coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ in V is equal to 1 (it comes from the product of terms on the diagonal of the matrix) and the coefficient in the product is also equal to 1 (it comes from the product of the all the first terms x_j with $j > i$ in each pair), the constant must equal 1.
   - Thus, $V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$ as claimed.

   (e) Suppose that $x_1, \ldots, x_n \in F$ are distinct and $y_1, \ldots, y_n \in F$ are arbitrary. Prove that there exists a unique polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ in $F[x]$ of degree at most $n - 1$ such that $p(x_i) = y_i$ for each $1 \leq i \leq n$. [Hint: Write down the corresponding system of linear equations.]

   - We have the equations $a_0 + a_1 x_1 + \cdots + a_{n-1}x_1^{n-1} = y_1$, ... , $a_0 + a_1 x_n + \cdots + a_{n-1}x_n^{n-1} = y_n$, which in matrix form is $\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$.
   - The coefficient matrix is precisely the Vandermonde matrix we have been analyzing. By the formula in part (d), its determinant is nonzero (as all of the x_i are distinct) and therefore it is invertible, so the system has a unique solution.
   - This means there is a unique solution to the system, which is to say, there is a unique polynomial $p(x)$ with the desired properties.
