E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 2, due Tue Jan 20th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:

   (a) There is a system of linear equations over $\mathbb{R}$ having exactly two different solutions.

   (b) If A is $n \times n$ and $\mathbf{c}$ is $n \times 1$, then the matrix system $A\mathbf{x} = \mathbf{c}$ always has a unique solution for $\mathbf{x}$.

   (c) Every vector space contains a zero vector.

   (d) In any vector space, $\alpha \mathbf{v} = \alpha \mathbf{w}$ implies that $\mathbf{v} = \mathbf{w}$.

   (e) In any vector space, $\alpha \mathbf{v} = \beta \mathbf{v}$ implies that $\alpha = \beta$.

   (f) If U is a subspace of V and V is a subspace of W, then U is a subspace of W.

   (g) The empty set is a subspace of any vector space.

   (h) The intersection of two subspaces is always a subspace.

   (i) The union of two subspaces is always a subspace.

   (j) The union of two subspaces is never a subspace.

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2. Determine whether or not each given set S is a subspace of the given vector space V. For each set that is not a subspace, identify at least one part of the subspace criterion that fails.

   (a) $V = \mathbb{R}^4$, $S =$ the vectors $\langle a, b, c, d \rangle$ in $\mathbb{R}^4$ with $2a + 0b + 2c + 6d = 2026$.

   (b) $V = \mathbb{R}^4$, $S =$ the vectors $\langle a, b, c, d \rangle$ in $\mathbb{R}^4$ with $abcd = 0$.

   (c) $V = \mathbb{R}^4$, $S =$ the vectors $\langle a, b, c, d \rangle$ with $d = a + b + c$ and $a = c$.

   (d) $V =$ differentiable functions on $[0, 1]$, $S =$ the functions with $f'(x) = f(x)$.

   (e) $V =$ real-valued functions on $\mathbb{R}$, $S =$ the functions with $f(x) = f(1 - x)$ for all real x.

   (f) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with integer entries.

   (g) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with nonnegative real entries.

   (h) $V = P_5(\mathbb{R})$, $S =$ the polynomials in V divisible by x^2.

   (i) $V = P_3(\mathbb{C})$, $S =$ the polynomials in V with $p(i) = 0$.

   (j) $V = M_{2\times 2}(\mathbb{Q})$, $S =$ the matrices in V of determinant zero.

   (k) $V =$ real-valued functions on $\mathbb{R}$, $S =$ the functions that are zero at every rational number.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Show the following things, by induction or otherwise:

   (a) Prove that $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = (n-1)n(n+1)/3$ for all integers $n \geq 2$.

   (b) Prove that the nth power of the matrix $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix}$ for each positive integer n.

   (c) Prove that a matrix with m rows can be put into row-echelon form using at most $m(m-1)/2$ elementary row operations. [Hint: If a matrix has $m+1$ rows, explain why it takes at most m row operations to clear out the entries in the first column.]

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4. The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Thus, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and so forth. By convention we also set $F_0 = 0$, satisfying the same recurrence.

   (a) Prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ for every positive integer n.

   (b) Prove that the nth power of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ for each positive integer n.

   (c) Prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for each positive integer n. [Hint: Use (b).]

   (d) Let M_n be the $n \times n$ matrix with 1s on the diagonal and directly below the diagonal, -1s directly above the diagonal, and 0s elsewhere. Prove that $\det(M_n)$ is the $(n+1)$st Fibonacci number F_{n+1}.

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5. Suppose A is an $m \times n$ matrix with entries from the field F.

   (a) Show that the set of all vectors $\mathbf{x} \in F^n$ such that $A\mathbf{x}$ equals the zero vector (in F^m) is a subspace of F^n.

   (b) Deduce that the set of solutions to any homogeneous system of linear equations (i.e., in which all of the constants are equal to zero) over F is an F-vector space.

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6. If V is a vector space and W_1, W_2 are two subspaces of V, their <u>sum</u> is defined to be the set $W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 : \mathbf{w}_1 \in W_1 \text{ and } \mathbf{w}_2 \in W_2\}$ of all sums of an element of W_1 with an element of W_2.

   (a) Prove that $W_1 + W_2$ contains W_1 and W_2, and is a subspace of V.

   (b) Show that if W is a subspace of V containing W_1 and W_2, then W must contain $W_1 + W_2$. Deduce that $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2.

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7. [Challenge] Let F be a field and suppose $x_1, \ldots, x_n$ are elements of F. The goal of this problem is to evaluate the famous <u>Vandermonde determinant</u> $V(x_1, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$.

   (a) Show that if any of the x_i are equal to one another, then $V(x_1, \ldots, x_n) = 0$.

   (b) Show that as a polynomial in the variables $x_1, \ldots, x_n$, $V(x_1, \ldots, x_n)$ has degree $\frac{n(n-1)}{2}$ and is divisible by $x_j - x_i$ for any $i \neq j$. [Hint: Use (a) and the remainder theorem.]

   (c) Deduce that $V(x_1, \ldots, x_n)$ is divisible by the product $\prod_{1 \leq i < j \leq n}(x_j - x_i)$ and that this product is a polynomial of degree $\frac{n(n-1)}{2}$.

   (d) Show in fact that $V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$. [Hint: Compare degrees and coefficients of $x_1^0 x_2^1 \cdots x_n^{n-1}$ on both sides.]

   (e) Suppose that $x_1, \ldots, x_n \in F$ are distinct and $y_1, \ldots, y_n \in F$ are arbitrary. Prove that there exists a unique polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ in $F[x]$ of degree at most $n-1$ such that $p(x_i) = y_i$ for each $1 \leq i \leq n$. [Hint: Write down the corresponding system of linear equations.]
