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1. Identify each of the following statements as true or false:

   (a) The set of integers $\mathbb{Z}$ is not a field.

   - $\boxed{\text{True}}$. Some elements, like 2 and 5, have no multiplicative inverse in $\mathbb{Z}$.

   (b) Every field has infinitely many elements.

   - $\boxed{\text{False}}$. There exist fields with finitely many elements, like $\mathbb{Z}/p\mathbb{Z}$.

   (c) It is impossible to have $6 = 0$ in a field F.

   - $\boxed{\text{False}}$. There exist fields where $6 = 0$, like the fields $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

   (d) For any $n \times n$ matrices A and B, $(A + B)^2 = A^2 + 2AB + B^2$.

   - $\boxed{\text{False}}$. The correct formula would be $(A + B)^2 = A^2 + AB + BA + B^2$, since matrix multiplication is not commutative.

   (e) For any $n \times n$ matrices A and B, $(BA)^T = B^T A^T$.

   - $\boxed{\text{False}}$. The correct formula would be $(BA)^T = A^T B^T$.

   (f) For any invertible $n \times n$ matrices A and B, $(A + B)^{-1} = A^{-1} + B^{-1}$.

   - $\boxed{\text{False}}$. In fact this formula is almost never correct (see problem 6c). An explicit counterexample is $A = B = I_n$: then $(A + B)^{-1} = \frac{1}{2} I_n$ while $A^{-1} + B^{-1} = 2I_n$.

   (g) For any invertible $n \times n$ matrices A and B, $(BA)^{-1} = A^{-1}B^{-1}$.

   - $\boxed{\text{True}}$. The inverse of a product is the product of the inverses in reverse order.

   (h) If A and B are $n \times n$ matrices with $\det(A) = 2$ and $\det(B) = 3$, then $\det(AB) = 6$.

   - $\boxed{\text{True}}$. The determinant is multiplicative so $\det(AB) = \det(A)\det(B) = 6$.

   (i) If A is an $n \times n$ matrix with $\det(A) = 3$, then $\det(2A) = 3n$.

   - $\boxed{\text{False}}$. Doubling a matrix doubles each row, so if there are n rows, the correct formula would be $\det(2A) = 2^n \cdot 3$.

   (j) For any $n \times n$ matrix A, $\det(A) = -\det(A^T)$.

   - $\boxed{\text{False}}$. The determinant of a transpose equals the determinant of the original matrix, so $\det(A) = \det(A^T)$.

   (k) For any $n \times n$ matrices A and B, $\det(AB) = \det(B)\det(A)$.

   - $\boxed{\text{True}}$. The determinant is multiplicative so $\det(AB) = \det(A)\det(B) = \det(B)\det(A)$.

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2. Find the general solution to each system of linear equations:

   (a) $\left\{ \begin{array}{c} -x - 3y + 5z = 9 \\ 3x + 2y + 2z = 0 \\ 2x + 2y + 3z = 4 \end{array} \right\}$.

   - By row-reducing, the solution is $(x, y, z) = \boxed{(-2, 1, 2)}$.

   (b) $\left\{ \begin{array}{c} x - 2y + 4z = 4 \\ 2x + 4y + 8z = 0 \end{array} \right\}$.

   - By row-reducing the solution is $(x, y, z) = \boxed{(2 - 4z, -1, z)}$.

(c) $\left\{\begin{array}{l} a+b+c+d \quad\quad = 2 \\ a+b+c \quad\quad +e = 3 \\ a+b \quad\quad +d+e = 4 \\ a \quad\quad +c+d+e = 5 \\ \quad\quad b+c+d+e = 6 \end{array}\right\}$.

- By row-reducing, the solution is $\boxed{(a,b,c,d,e) = (-1,0,1,2,3)}$.

(d) $\left\{\begin{array}{l} x+3y+\ \ z = -4 \\ -x-6y+8z = 10 \\ 2x+4y+8z = 0 \end{array}\right\}$.

- By row-reducing, there is $\boxed{\text{no solution}}$.

(e) $\left\{\begin{array}{rcl} a+b+c+d+e & = & 1 \\ a+2b+3c+4d+5e & = & 6 \end{array}\right\}$.

- By row-reducing, the solution is $(a,b,c,d,e) = \boxed{(-4+c+2d+3e, 5-2c-3d-4e)}$.

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3. Compute the following things:

(a) The reduced row-echelon forms of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 & 2 & 3 \\ 2 & 1 & 0 & -1 & -2 \\ -4 & -2 & 0 & 3 & 0 \end{bmatrix}$.

- Row-reducing yields the RREFs $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) The determinants of $\begin{bmatrix} -1 & 5 & 2 \\ 0 & -3 & 7 \\ 2 & 8 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \\ 8 & 27 & 64 & 125 \end{bmatrix}$.

- The determinants are $\det(A) = 141$ and $\det(B) = 12$.

(c) The inverses of $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -3 & -2 \\ -3 & 7 & 8 \\ 2 & -6 & -5 \end{bmatrix}$.

- The inverses are $\begin{bmatrix} -3 & -3 & 1 \\ -1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ and $\frac{1}{2}\begin{bmatrix} 13 & -3 & -10 \\ 1 & -1 & -2 \\ 4 & 0 & -2 \end{bmatrix}$ respectively.

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4. Suppose that A and B are $n \times n$ matrices with entries from a field F.

(a) If AB is invertible, show that A and B are invertible.

- Notice that $AB(AB)^{-1} = I_n$, and so $B(AB)^{-1}$ is a right inverse of the matrix A. This means A is invertible.
- Likewise, $(AB)^{-1}AB = I_n$, so $(AB)^{-1}A$ is a left inverse of the matrix B. This means B is invertible.
- Alternatively, since AB is invertible, $\det(AB) = \det(A)\det(B)$ is nonzero. This can only happen when $\det(A)$ and $\det(B)$ are both nonzero, which is to say, when A and B are both invertible.

(b) If A is invertible, show that A^T is invertible and that its inverse is $(A^{-1})^T$.

- Since $\det(A^T) = \det(A)$, if A is invertible then A^T will also be invertible.
- Furthermore, by using the fact that $A^T B^T = (BA)^T$ with $B = A^{-1}$, we see that $A^T(A^{-1})^T = (A^{-1}A)^T = (I_n)^T = I_n$.

- In the same way, $(A^{-1})^T A^T = (AA^{-1})^T = (I_n)^T = I_n$, and so $(A^{-1})^T$ satisfies the inverse matrix property for A^T: this means $(A^T)^{-1} = (A^{-1})^T$.

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5. Let F be a field of characteristic not 2 (i.e., in which $2 \neq 0$). A square matrix A with entries from F is called symmetric if $A = A^T$ and skew-symmetric if $A = -A^T$.

   (a) For any $n \times n$ matrix B, show that $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric.

   - Observe that $(B + B^T)^T = B^T + (B^T)^T = B^T + B$ so this matrix equals its transpose hence is symmetric.
   - Similarly, $(B - B^T)^T = B^T - B$, so this matrix is -1 times its transpose hence is skew-symmetric.

   (b) Show that any square matrix M can be written *uniquely* in the form $M = S + T$ where S is symmetric and T is skew-symmetric. [Make sure to prove that there is *only* one such decomposition!]

   - If $M = S + T$ then $M^T = S^T + T^T = S - T$. Solving for S, T produces $S = \frac{1}{2}(M + M^T)$ and $T = \frac{1}{2}(M - M^T)$, so this is the only possible solution. (Here we are using the fact that $2 \neq 0$, so we can divide by 2.)
   - By part (a), we see $S = \frac{1}{2}(M + M^T)$ is symmetric and $T = \frac{1}{2}(M - M^T)$ is skew-symmetric, so these choices do work. Hence there is a unique decomposition as claimed.

   (c) If A is a skew-symmetric $n \times n$ real matrix and n is odd, show that $\det(A) = 0$.

   - Taking the determinant of both sides of $\det(A) = \det(-A^T)$ yields $\det(A) = (-1)^n \det(A^T) = (-1)^n \det(A)$.
   - Since n is odd, this gives $\det(A) = -\det(A)$, meaning $\det(A) = 0$ since $2 \neq 0$ (and thus $1 \neq -1$).

   (d) If A and B are symmetric, prove that AB is symmetric if and only if A and B commute (i.e., $AB = BA$).

   - We have $(AB)^T = B^T A^T = BA$, and so we see $AB = (AB)^T$ if and only if $AB = BA$, as desired.

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6. The goal of this problem is to prove a matrix inversion formula called the Woodbury matrix identity, and then give an application.

   (a) Suppose P and Q are $n \times n$ matrices such that $I_n + QP$ is invertible. Show that $I_n + PQ$ is also invertible and that its inverse is $M = I_n - P(I_n + QP)^{-1}Q$. [Hint: Multiply out $M(I_n + PQ)$.]

   - Let $M = I_n - P(I_n + QP)^{-1}Q$. Then $M(I_n + PQ) = (I_n + PQ) - P(I_n + QP)^{-1}Q(I_n + PQ) = (I_n + PQ) - P(I_n + QP)^{-1}(Q + QPQ) = (I_n + PQ) - P(I_n + QP)^{-1}(I_n + QP)Q = (I_n + PQ) - PQ = I_n$ and therefore M is the inverse of $I_n + PQ$.

   (b) Prove the Woodbury matrix identity: if A is an invertible $n \times n$ matrix, U is an $n \times k$ matrix, C is an invertible $k \times k$ matrix, and V is a $k \times n$ matrix such that $C^{-1} + VA^{-1}U$ is invertible, then $A + UCV$ is invertible and

   $$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$

   [Hint: Observe $A + UCV = A(I_n + PQ)$ where $P = A^{-1}U$ and $Q = CV$, then use (a).]

   - Following the hint, first notice that $A + UCV = A(I_n + A^{-1}UCV) = A(I_n + PQ)$ where by setting $P = A^{-1}U$ and $Q = CV$.
   - Taking the inverse of both sides yields $(A + UCV)^{-1} = (I_n + PQ)^{-1}A^{-1}$, and now we can use the identity from part (a).
   - We need to check that $I_n + QP = I_n + CVA^{-1}U = C(C^{-1} + VA^{-1}U)$ is invertible, which is true because it is the product of the two invertible matrices C and $C^{-1}VA^{-1}U$.
   - Then (a) yields that $I_n + PQ = I_n + A^{-1}UCV$ is invertible and its inverse is $I_n - P(I_n + QP)^{-1}Q = I_n - A^{-1}U(I_n + CVA^{-1}U)^{-1}CV$.

- To get the desired identity, we can observe that $(I_n + CVA^{-1}U)^{-1}C = (C^{-1} + C^{-1}CVA^{-1}U)^{-1} = (C^{-1} + VA^{-1}U)^{-1}$, and then finally plug in to get $(A + UCV)^{-1} = (I_n + PQ)^{-1}A^{-1} = [I_n - A^{-1}U(I_n + CVA^{-1}U)^{-1}CV]A^{-1} = A^{-1} - A^{-1}U(I_n + CVA^{-1}U)^{-1}CV]A^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$ as desired.

(c) Suppose A and C are invertible $n \times n$ matrices and $A + C$ is also invertible. Show that $(A + C)^{-1} = A^{-1} - (A + AC^{-1}A)^{-1}$.

- Apply the Woodbury matrix identity with $U = V = I_n$: this yields $(A + C)^{-1} = A^{-1} - A^{-1}(C^{-1} + A^{-1})^{-1}A^{-1} = A^{-1} - (AC^{-1}A + A)^{-1}$ after combining the terms on the right.

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7. [Challenge] Let D_n denote the value of the $(n-1) \times (n-1)$ determinant $\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$.

Determine whether $\lim_{n\to\infty} \dfrac{D_n}{n!}$ exists. [Hint: Start by subtracting the first row from the other rows.]

- Subtract the first row from the other rows, yielding $\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ -2 & 3 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 4 & 0 & \cdots & 0 \\ -2 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & 0 & \cdots & n \end{vmatrix}$.
- Now subtract 1/3 of the second row, 1/4 of the third row, 1/5 of the fourth row, ... , $1/n$th of the $(n-1)$st row, from the first row.
- This yields $\begin{vmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ -2 & 3 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 4 & 0 & \cdots & 0 \\ -2 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & 0 & \cdots & n \end{vmatrix}$ where $x = 3 + \dfrac{2}{3} + \dfrac{2}{4} + \dfrac{2}{5} + \cdots + \dfrac{2}{n-1}$.
- Now the matrix is lower triangular so its determinant is simply $x \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot n$.
- Then $\dfrac{D_n}{n!} = \dfrac{x}{2} = \dfrac{3}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \cdots + \dfrac{1}{n-1} = \sum_{k=1}^{n-1} \dfrac{1}{k}$. As $n \to \infty$ this series is the harmonic series, which $\boxed{\text{diverges to } \infty}$.
- Remark: This was problem B5 from the 1992 Putnam.
