1. Identify each of the following statements as true or false:

   (a) The function $Q(x, y) = xy$ on $\mathbb{R}^2$ is a quadratic form.

   - $\boxed{\text{True}}$: all of the terms have total degree 2, so it is a quadratic form.

   (b) The function $Q(x, y, z) = x^2 - 4xy + xyz + z^2$ on $\mathbb{R}^3$ is a quadratic form.

   - $\boxed{\text{False}}$: the term xyz has total degree 3 so this is not a quadratic form.

   (c) The function $Q(f) = \int_0^1 x\, f(x)^2\, dx$ on $\mathbb{R}[x]$ is a quadratic form.

   - $\boxed{\text{True}}$: it is the quadratic form associated to the bilinear form $\Phi(f, g) = \int_0^1 x\, f(x)g(x)\, dx$ on $\mathbb{R}[x]$.

   (d) Every quadratic form over $\mathbb{R}$ is a bilinear form.

   - $\boxed{\text{False}}$: although quadratic forms and bilinear forms have a relationship to one another, they are not the same thing. Quadratic forms are a function on V while bilinear forms are a function on $V \times V$.

   (e) Every quadratic form over an arbitrary field is a bilinear form.

   - $\boxed{\text{False}}$: this is still false. (This one was intended to make you second-guess your answer to (d).)

   (f) The second derivatives test classifies any critical point as a local minimum, local maximum, or saddle.

   - $\boxed{\text{False}}$: there are situations in which the second derivatives test is inconclusive, namely, when zero is an eigenvalue and all the other eigenvalues have the same sign.

   (g) If both eigenvalues of the 2×2 real symmetric matrix S are positive, then the graph of $(x, y)\cdot S\cdot(x, y)^T = 1$ in $\mathbb{R}^2$ will be an ellipse.

   - $\boxed{\text{True}}$: if we diagonalize the quadratic form, then in the new coordinate system the graph will be of the form $a(x')^2 + b(y')^2 = 1$ with $a, b > 0$, and this is an ellipse.

   (h) If one eigenvalue of the 2×2 real symmetric matrix S is zero and the other is nonzero, then the graph of $(x, y) \cdot S \cdot (x, y)^T = 1$ in $\mathbb{R}^2$ will be a hyperbola.

   - $\boxed{\text{False}}$: if we diagonalize the quadratic form, then in the new coordinate system the graph will be of the form $a(x')^2 = 1$ with $a \neq 0$, and this is actually a pair of lines.

   (i) The singular values of $T : V \to V$ are the absolute values of the eigenvalues of T.

   - $\boxed{\text{False}}$: they are the square roots of the eigenvalues of T^*T, which are generally different.

   (j) If T is Hermitian, the singular values of $T : V \to V$ are absolute values of the eigenvalues of T.

   - $\boxed{\text{True}}$: by the spectral mapping theorem, the eigenvalues of $T^*T = T^2$ are the squares of the eigenvalues of T. Then the singular values are the nonnegative square roots of these, which are just their absolute values since the eigenvalues are real by the spectral theorem.

   (k) The singular value decomposition of a matrix is unique.

   - $\boxed{\text{False}}$: there are always many possible choices of orthonormal bases of unit eigenvectors for V (for example, we could scale the eigenvectors by -1).

   (l) If $T : V \to W$ is linear, the pseudoinverse $T^\dagger$ satisfies $TT^\dagger(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \operatorname{im}(T)$.

   - $\boxed{\text{True}}$: $T^\dagger$ behaves as the inverse of T on $\operatorname{im}(T)$.

   (m) If $T : V \to W$ is linear, the pseudoinverse $T^\dagger$ satisfies $TT^\dagger(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \operatorname{im}(T)^\perp$.

   - $\boxed{\text{False}}$: $T^\dagger(\mathbf{w}) = \mathbf{0}$ for $\mathbf{w} \in \operatorname{im}(T)^\perp$.

   (n) If $T : V \to V$ is an isomorphism, then $T^\dagger = T^{-1}$.

   - $\boxed{\text{True}}$: when T is invertible, the pseudoinverse is simply the inverse of T.

2. Consider the bilinear form $\Phi[(a,b),(c,d)] = 4ac - 2ad - 2bc + 7bd$ on $\mathbb{R}^2$ with associated quadratic form Q.

(a) Write down Q explicitly and also find $[\Phi]_\beta$ for $\beta = \{(1,0),(0,1)\}$.

- We have $Q(x,y) = \Phi[(x,y),(x,y)] = \boxed{4x^2 - 4xy + 7y^2}$ and $[\Phi]_\beta = \boxed{\begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}}$.

(b) Find an orthonormal basis γ for $\mathbb{R}^2$ such that $[\Phi]_\gamma$ is diagonal, and compute the diagonalization $[\Phi]_\gamma$.

- We compute the eigenvalues and eigenvectors of $[\Phi]_\beta$: the characteristic polynomial is $p(t) = (t-4)(t-7) - 4 = t^2 - 11t + 24 = (t-3)(t-8)$, so the eigenvalues are $\lambda = 3, 8$.
- We can compute eigenvectors $(2,1)$ and $(-1,2)$ for $\lambda = 3, 8$ respectively, so upon normalizing these eigenvectors, we see that we can take $\gamma = \boxed{\left\{\frac{1}{\sqrt{5}}(2,1), \frac{1}{\sqrt{5}}(-1,2)\right\}}$
- Then, with $Q = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, we have $[\Phi]_\gamma = Q^T[\Phi]_\beta Q = \boxed{\begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}}$.

(c) Describe the shape of the quadratic variety $Q(x,y) = 1$ in $\mathbb{R}^2$ as one of the 3 standard conic sections.

- In the rotated coordinate system described in part (b), the quadratic variety $Q(x,y)$ has equation $3(x')^2 + 8(y')^2 = 1$, so it is an $\boxed{\text{ellipse}}$.
- In fact, because we have used an orthogonal change of coordinates, we can even find the lengths of the semimajor and semiminor axes of the ellipse: they are $1/\sqrt{3}$ and $1/\sqrt{8}$.

(d) Classify the critical point of $Q(x,y)$ at $(0,0)$ as a local minimum, local maximum, or saddle point.

- We simply apply the second derivatives test: the Hessian matrix is simply $[\Phi]_\beta$ so from our calculations in part (b), the eigenvalues are 3 and 8.
- Therefore, since all the eigenvalues are positive, the critical point at $(0,0)$ is a $\boxed{\text{local minimum}}$.

(e) Calculate the signature and index of Q, and determine the definiteness of Q.

- There are 2 positive eigenvalues and 0 negative eigenvalues, so the signature is $2 - 0 = \boxed{2}$ and the index is also $\boxed{2}$.
- Since all eigenvalues are positive (hence also nonnegative), Q is $\boxed{\text{positive definite}}$.

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3. Consider the quadratic form $Q(x,y,z) = 11x^2 + 40xy - 16xz - 16y^2 - 16yz + 5z^2$ on $\mathbb{R}^3$.

(a) Find the symmetric matrix S associated to the underlying bilinear form for Q with respect to the standard basis $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$.

- We just read off the entries from the coefficients, yielding $S = \boxed{\begin{bmatrix} 11 & 20 & -8 \\ 20 & -16 & -8 \\ -8 & -8 & 5 \end{bmatrix}}$.

(b) Give an explicit orthonormal change of basis that diagonalizes Q, and find the resulting diagonalization.

- We compute the eigenvalues and eigenvectors of S: the characteristic polynomial is $p(t) = t^3 - 729t = t(t-27)(t+27)$, so the eigenvalues are $\lambda = 27, -27, 0$.
- We can compute eigenvectors $(7,4,-4)$, $(4,-8,-1)$, and $(4,1,8)$, for $\lambda = 27, -27, 0$ respectively.
- Upon normalizing, for $Q = \dfrac{1}{9}\begin{bmatrix} 7 & 4 & 4 \\ 4 & -8 & 1 \\ -4 & -1 & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 27 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we have $Q^T S Q = D$.
- Then the desired change of basis is $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = Q \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, or explicitly, $\boxed{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} (7x+4y+4z)/9 \\ (4x-8y+z)/9 \\ (-4x-y+8z)/9 \end{bmatrix}}$,

  and the resulting diagonalization is $Q(x,y,z) = \boxed{27(x')^2 - 27(y')^2}$.

(c) Describe the shape of the quadratic variety $Q(x, y, z) = 1$ in $\mathbb{R}^3$ as one of the 9 standard quadric surfaces.

- From the calculations in part (b), after applying the orthonormal change of basis, the new equation for the variety is $27(x')^2 - 27(y')^2 = 1$.
- Upon rescaling by the factor of 27, this has the form $x^2 - y^2 = 1$, which is a $\boxed{\text{hyperbolic cylinder}}$.

(d) Classify the critical point of $Q(x, y, z)$ at $(0, 0, 0)$ as a local minimum, local maximum, or saddle point.

- We simply apply the second derivatives test: the Hessian matrix is simply S so from our calculations in part (b), the eigenvalues are 27, −27, and 0.
- Therefore, since there is both a positive eigenvalue and a negative eigenvalue, the critical point is a $\boxed{\text{saddle point}}$.

(e) Calculate the signature and index of Q, and determine the definiteness of Q.

- There is 1 positive eigenvalue and 1 negative eigenvalue, so the signature is $1 - 1 = \boxed{0}$ and the index is $\boxed{1}$.
- Since there is both a positive and a negative eigenvalue, the quadratic form is $\boxed{\text{indefinite}}$.

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4. For each matrix M, find (i) the singular values of M, (ii) a singular value decomposition $M = U\Sigma V^*$ where U and V are unitary and Σ is a rectangular diagonal matrix, and (iii) the pseudoinverse $M^\dagger$ of M:

(a) $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$.

- We have $M^*M = \begin{bmatrix} 20 & 22 \\ 22 & 53 \end{bmatrix}$ with eigenvalues $\lambda = 64, 9$ yielding singular values $\sigma_1 = \boxed{8}$, $\sigma_2 = \boxed{3}$.
- We can also compute an orthonormal eigenbasis with corresponding eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{M\mathbf{v}_2}{\sigma_2} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
- Thus we obtain $U = \boxed{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}$, $\Sigma = \boxed{\begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}}$, $V^* = \boxed{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}$.
- Since M is invertible, its pseudoinverse is simply its inverse $M^{-1} = \boxed{\frac{1}{24}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}}$. (Alternatively, we could write it out using the SVD formula.)

(b) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$.

- We have $M^*M = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$ with eigenvalues $\lambda = 85, 0$ yielding nonzero singular value $\sigma_1 = \boxed{\sqrt{85}}$.
- We can also compute an orthonormal eigenvasis with corresponding eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and completing to an orthonormal basis gives $\mathbf{w}_2 = \frac{1}{\sqrt{17}}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$.
- Thus $U = \boxed{\frac{1}{\sqrt{17}}\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}}$, $\Sigma = \boxed{\begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix}}$, $V^* = \boxed{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}$.
- Using the pseudoinverse formula yields $M^\dagger = V\Sigma^\dagger U^* = \boxed{\frac{1}{85}\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}}$.

(c) $\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & 6 \end{bmatrix}$.

- We have $M^*M = \begin{bmatrix} 8 & -2 & 6 \\ -2 & 5 & 12 \\ 6 & 12 & 45 \end{bmatrix}$ with eigenvalues $\lambda = 49, 9, 0$ yielding nonzero singular values $\sigma_1 = \boxed{7}$, $\sigma_2 = \boxed{3}$.
- We also compute unit eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{245}} \begin{bmatrix} 2 \\ 4 \\ 15 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{7} \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$. Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{M\mathbf{v}_2}{\sigma_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Thus $U = \boxed{\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}$, $\Sigma = \boxed{\begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}}$, $V^* = \boxed{\begin{bmatrix} 2/\sqrt{245} & -2/\sqrt{5} & 3/7 \\ 4/\sqrt{245} & 1/\sqrt{5} & 6/7 \\ 15/\sqrt{245} & 0 & -2/7 \end{bmatrix}}$.
- Using the pseudoinverse formula yields $M^\dagger = V\Sigma^\dagger U^* = \boxed{\frac{1}{147} \begin{bmatrix} -38 & 22 \\ 22 & -5 \\ 9 & 18 \end{bmatrix}}$.

(d) $\begin{bmatrix} -2 & 2 \\ 2 & 1 \\ 3 & 6 \end{bmatrix}$.

- This is the adjoint of the matrix in (c) so the SVD just has U and V swapped and Σ and $M^\dagger$ transposed:
$U = \boxed{\begin{bmatrix} 2/\sqrt{45} & -2/\sqrt{5} & -1/3 \\ 4/\sqrt{45} & 1/\sqrt{5} & -2/3 \\ 5/\sqrt{45} & 0 & 2/3 \end{bmatrix}}$, $\Sigma = \boxed{\begin{bmatrix} 7 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}}$, $V = \boxed{\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}$, $M^\dagger = \boxed{\frac{1}{147} \begin{bmatrix} -38 & 22 & 9 \\ 22 & -5 & 18 \end{bmatrix}}$

(e) $\begin{bmatrix} 1 & i & -1 & -i \\ 2 & 2 & 2 & 2 \end{bmatrix}$.

- We have $M^*M = \begin{bmatrix} 5 & 4+i & 3 & 4-i \\ 4-i & 5 & 4+i & 3 \\ 3 & 4-i & 5 & 4+i \\ 4+i & 3 & 4-i & 5 \end{bmatrix}$ with eigenvalues $\lambda = 16, 4, 0, 0$ yielding nonzero singular values $\sigma_1 = \boxed{4}$, $\sigma_2 = \boxed{2}$.
- We find orthonormal eigenvectors $\mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} -i \\ -1 \\ i \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{2} \begin{bmatrix} -1+i \\ -i \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 \\ -1-i \\ 2 \\ -1+i \end{bmatrix}$.
Then $\mathbf{w}_1 = \frac{M\mathbf{v}_1}{\sigma_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \frac{M\mathbf{v}_2}{\sigma_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus $U = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$, $\Sigma = \boxed{\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}}$,
$V^* = \boxed{\frac{1}{2} \begin{bmatrix} 1 & -i & -1+i & 0 \\ 1 & -1 & -i & (-1-i)/\sqrt{2} \\ 1 & i & 0 & \sqrt{2} \\ 1 & 1 & 1 & (-1+i)/\sqrt{2} \end{bmatrix}}$.
- Using the pseudoinverse formula yields $M^\dagger = V\Sigma^\dagger U^* = \boxed{\frac{1}{8} \begin{bmatrix} 2 & 1 \\ -2i & 1 \\ -2 & 1 \\ 2i & 1 \end{bmatrix}}$.

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5. Let $A = \begin{bmatrix} 2 & -8 & 2 \\ 6 & 6 & -9 \end{bmatrix}$.

(a) Find singular value decompositions for A and for A^T.

- We compute $A^*A = A^TA = \begin{bmatrix} 40 & 20 & -50 \\ 20 & 100 & -70 \\ -50 & -70 & 85 \end{bmatrix}$ with eigenvalues $\lambda = 180, 45, 0$ and orthonormal eigenvectors $\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.
- The nonzero singular values are $\sigma_1 = \sqrt{180}$ and $\sigma_2 = \sqrt{45}$ and then $\mathbf{w}_1 = A\mathbf{v}_1/\sigma_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{w}_2 = A\mathbf{v}_2/\sigma_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ -1 \end{bmatrix}$.
- Thus $U = \boxed{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}}$, $\Sigma = \boxed{\begin{bmatrix} \sqrt{180} & 0 & 0 \\ 0 & \sqrt{45} & 0 \end{bmatrix}}$, $V^* = V = \boxed{\frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}}$.

(b) Find the pseudoinverses $A^\dagger$ and $(A^T)^\dagger$.

- Using the pseudoinverse formula yields $A^\dagger = V\Sigma^\dagger U^* = \boxed{\frac{1}{90}\begin{bmatrix} 7 & 6 \\ -10 & 0 \\ -2 & -6 \end{bmatrix}}$ and then $(A^T)^\dagger = (A^\dagger)^T = \boxed{\frac{1}{90}\begin{bmatrix} 7 & -10 & -2 \\ -6 & 0 & -6 \end{bmatrix}}$.

(c) Find the solution $\mathbf{x}$ to the system $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ of minimal norm.

- By our results on pseudoinverses, this is $\hat{\mathbf{x}} = A^\dagger \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2/9 \\ -2/9 \\ -1/9 \end{bmatrix}}$.

(d) Find the least-squares solution to the inconsistent system $A^T\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

- By our results on pseudoinverses, this is $\hat{\mathbf{x}} = (A^T)^\dagger = (A^\dagger)^T \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1/10 \\ -1/5 \end{bmatrix}}$.

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6. In multivariable calculus, the following more explicit version of the second derivative test is often taught[1]:

- Theorem (Second Derivatives Test in $\mathbb{R}^2$): Suppose P is a critical point of $f(x, y)$, and let D be the value of the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at P. If $D > 0$ and $f_{xx} > 0$, then the critical point is a minimum. If $D > 0$ and $f_{xx} < 0$, then the critical point is a maximum. If $D < 0$, then the critical point is a saddle point. If $D = 0$, then the test is inconclusive.

Using our general version of the second derivatives test, prove this variation. [Hint: Note $D = \det(H) = \lambda_1\lambda_2$.]

- We want to compute the eigenvalues of the Hessian matrix, which in this case is $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.
- As noted in the hint, we have $D = \det(H) = f_{xx}f_{yy} - f_{xy}^2$.

[1]The statement of this theorem is copied directly from my multivariable calculus course notes, in fact!

- The eigenvalues are always real numbers, so by applying our general version of the test, we have a small list of possibilities: (i) both eigenvalues are positive (local minimum), (ii) both eigenvalues are negative (local maximum), (iii) one eigenvalue is positive and the other is negative (saddle point), or (iv) there is a zero eigenvalue (inconclusive).
- If $D > 0$ then we are either in case (i) or case (ii). In this case the eigenvalues both have the same sign, which will therefore also be the same sign as the trace $f_{xx} + f_{yy}$ of the matrix (since the trace is equal to $\lambda_1 + \lambda_2$).
- But since $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, we must have $f_{xx}f_{yy} > 0$ and so that means f_{xx} and f_{yy} also have the same sign, which is therefore also the common sign of λ_1 and λ_2.
- Therefore, case (i) occurs precisely when $D > 0$ and $f_{xx} > 0$, while case (ii) occurs precisely when $D > 0$ and $f_{xx} < 0$. If $D < 0$ then we must be in case (iii). Finally, if $D = 0$ then we can only be in case (iv).
- So in all four cases, we see that the test above correctly identifies the type of critical point, as required.

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7. By the singular value decomposition theorem, if $T : V \to W$ is a linear transformation of rank r, then there exist orthonormal bases $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of V and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ of W along with scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ with $T(\mathbf{v}_i) = \sigma_i \mathbf{w}_i$ for $1 \leq i \leq r$ and $T(\mathbf{v}_i) = \mathbf{0}$ for $i > r$.

   (a) Show that $T^*(\mathbf{w}_i) = \sigma_i \mathbf{v}_i$ for $1 \leq i \leq r$ and $T(\mathbf{w}_i) = \mathbf{0}$ for $i > r$. [Hint: Consider $[T]_\beta^\gamma$ and $[T^*]_\gamma^\beta$.]

   - Note that $[T]_\beta^\gamma$ is an $m \times n$ (quasi)diagonal matrix A with diagonal entries $\sigma_1, \dots, \sigma_r, 0, \dots, 0$.
   - Since both β and γ are orthonormal bases, by our results on associated matrices of adjoints, we see that $[T^*]_\gamma^\beta$ is the adjoint matrix A^*, which since A has nonnegative real entries is just the transpose A^T, which is the $m \times n$ (quasi)diagonal matrix with diagonal entries $\sigma_1, \dots, \sigma_r, 0, \dots, 0$.
   - Converting back to the statement about bases, this means $T^*(\mathbf{w}_i) = \sigma_i \mathbf{v}_i$ for $1 \leq i \leq r$ and $T(\mathbf{w}_i) = \mathbf{0}$ for $i > r$ as claimed.

   (b) Show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of eigenvectors for T^*T with corresponding eigenvalues $\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0$, and that $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ is a set of eigenvectors for TT^* with corresponding eigenvalues $\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0$.

   - By (a) we have $T^*T(\mathbf{v}_i) = T^*(\sigma_i \mathbf{w}_i) = \sigma_i T^*(\mathbf{w}_i) = \sigma_i^2 \mathbf{v}_i$ for $1 \leq i \leq r$ and $T^*T(\mathbf{v}_i) = 0\mathbf{v}_i$ for $i > r$ which yields the first statement.
   - Similarly, we also have $TT^*(\mathbf{w}_i) = T(\sigma_i \mathbf{v}_i) = \sigma_i T(\mathbf{v}_i) = \sigma_i^2 \mathbf{v}_i$ for $1 \leq i \leq r$ and $TT^*(\mathbf{w}_i) = 0\mathbf{w}_i$ for $i > r$ which yields the second statement.

   (c) Deduce that the nonzero eigenvalues of T^*T and TT^* are the same, and hence that the nonzero singular values of T and T^* are the same.

   - This follows immediately from (b), since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of eigenvectors for T^*T its nonzero eigenvalues are $\sigma_1^2, \dots, \sigma_r^2$, and similarly since $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ is a basis of eigenvectors for TT^* its nonzero eigenvalues are $\sigma_1^2, \dots, \sigma_r^2$ also.
   - Since the eigenvalues of T^*T are the singular values of T, and the eigenvalues of TT^* are the singular values of T^*, the nonzero singular values of T and T^* are the same.

   (d) Show that if $A \in M_{m \times n}(\mathbb{C})$, then the singular values of A and A^* are the same, and that if A^* has a singular value decomposition $A = U\Sigma V^*$ then A^* has a singular value decomposition $A^* = V\Sigma^T U^*$.

   - The first part follows immediately from (c) applied to the transformation $T : \mathbb{C}^m \to \mathbb{C}^n$ with $T(\mathbf{v}) = A\mathbf{v}$.
   - The second part follows by taking the adjoint of $A = U\Sigma V^*$, yielding $A^* = (U\Sigma V^*)^* = V^{**}\Sigma^* U^* = V\Sigma^T U^*$ since $\Sigma^* = \Sigma^T$ because Σ is real.

Remark: The results of this problem are useful in computing the SVD of a non-square matrix, since one may just find the nonzero eigenvalues and eigenvectors of the smaller of A^*A and $AA^*$to construct $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$, and then compute $\ker(A)$ to get $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ and $\ker(A^*)$ to get $\{\mathbf{w}_{r+1}, \dots, \mathbf{w}_m\}$.

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8. [Challenge] A Hermitian matrix A is said to be positive-definite if $\mathbf{v}^* A\mathbf{v} > 0$ for every $\mathbf{v} \neq \mathbf{0}$. The goal of this problem is to prove Sylvester's criterion for positive-definiteness: if A is an $n \times n$ Hermitian matrix, then A is positive definite if and only if $\det A^{(k)} > 0$ for all $1 \le k \le n$, where $A^{(k)}$ is the upper $k \times k$ submatrix of A. So suppose $A \in M_{n\times n}(\mathbb{C})$ is Hermitian.

   (a) If A is positive definite, show that $A^{(k)}$ is positive definite for each $1 \le k \le n$ and deduce that $\det A^{(k)} > 0$ for all $1 \le k \le n$.

   - Suppose A is positive definite and let $\mathbf{v} = (x_1, \dots, x_k)$ where not all the x_i are zero.
   - For $\tilde{\mathbf{v}} = (x_1, \dots, x_k, 0, \dots, 0)$, we then have $\mathbf{v}^* A^{(k)}\mathbf{v} = \tilde{\mathbf{v}}^* A\tilde{\mathbf{v}} > 0$ since A is positive definite: since $\mathbf{v}$ is arbitrary, this means $A^{(k)}$ is positive definite.
   - Then $\det A^{(k)}$ is positive because it is the product of the eigenvalues of $A^{(k)}$ all of which are positive.

   (b) Suppose that A has two orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ whose eigenvalues λ_1, λ_2 are negative. Show that $A^{(k-1)}$ is not positive definite. [Hint: Show that there exists a linear combination $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ whose last coordinate is zero, and then that $\mathbf{w}^* A\mathbf{w} < 0$.]

   - If $\mathbf{v}_1$ or $\mathbf{v}_2$ has last coordinate zero simply take that vector itself, and otherwise if their last coordinates are x_1 and x_2 we may take $\mathbf{w} = x_2\mathbf{v}_1 - x_1\mathbf{v}_2$.
   - By the spectral theorem, $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Then $\mathbf{w}^* A\mathbf{w} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2)^* A(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) = (\overline{a_1}\mathbf{v}_1^* + \overline{a_2}\mathbf{v}_2^*)(a_1\lambda_1\mathbf{v}_1 + a_2\lambda_2\mathbf{v}_2) = \lambda_1 |a_1|^2 \mathbf{v}_1^*\mathbf{v}_1 + \lambda_2 |a_2|^2 \mathbf{v}_2^*\mathbf{v}_2 = \lambda_1 ||a_1||^2 + \lambda_2 ||a_2||^2$ where the two cross terms with $\mathbf{v}_1^*\mathbf{v}_2$ and $\mathbf{v}_2^*\mathbf{v}_1$ are zero by orthogonality.
   - Thus since $\lambda_1, \lambda_2 < 0$ and a_1, a_2 are not both zero, we see that $\mathbf{w}^* A\mathbf{w} < 0$. But since $\mathbf{w}^* A\mathbf{w} = \tilde{\mathbf{w}}^* A^{(k-1)}\tilde{\mathbf{w}}$ where $\tilde{\mathbf{w}}$ is obtained by dropping the 0 from the end of $\mathbf{w}$, this means $A^{(k-1)}$ is not positive definite.

   (c) Deduce that if $A^{(k-1)}$ is positive definite and $\det(A) > 0$, then all eigenvalues of A must be positive and hence A is positive definite.

   - By the spectral theorem, since A is Hermitian, its eigenvalues are all real and we may write down an orthonormal basis of eigenvectors.
   - Since $\det(A) > 0$ then because $\det(A)$ is the product of the eigenvalues of A, we see that A cannot have any zero eigenvalues and must have an even number of negative eigenvalues.
   - By the contrapositive of (c), we see that if $A^{(k-1)}$ is positive definite, then A cannot have two or more negative eigenvalues (since if it did, we could just take the associated orthonormal eigenvectors), so the only possibility is that it has no negative eigenvalues, in which case A is positive definite.

   (d) Suppose that $\det A^{(k)} > 0$ for all $1 \le k \le n$. Show that A is positive definite.

   - Induct on n. The base case $n = 1$ is trivial, since if the determinant of $[a]$ is positive, then $[a]$ is positive definite.
   - For the inductive step, if $\det A^{(k)} > 0$ for all $1 \le k \le n$, then applying the inductive hypothesis to $A^{(k-1)}$ shows that $A^{(k-1)}$ is positive definite.
   - Since we also have $\det(A) > 0$, by (c) this means A is positive definite, as claimed.

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9. [Extra Bonus Challenge] The goal of this problem is to discuss matrix square roots and the matrix analogue of the polar form $z = e^{i\theta}r$ of a complex number. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) Show that there exists a unitary matrix W and a positive semidefinite Hermitian matrix P such that $A = WP$; this is called a (right) polar decomposition of A with W being the analogue of $e^{i\theta}$ and P being the analogue of r. [Hint: Take $W = UV^*$ and $P = V\Sigma V^*$.]

   - Following the hint, let $A = U\Sigma V^*$ be a singular value decomposition of A.
   - Then the matrix $W = UV^*$ has $W^*W = (UV^*)^*(UV^*) = V^{**}U^*UV^* = VV^* = I_n$ so W is unitary.
   - Also, the matrix $P = V\Sigma V^*$ is unitarily equivalent (via V) to a diagonal matrix Σ with nonnegative diagonal entries, so it is a positive semidefinite Hermitian matrix.
   - Then $A = U\Sigma V^* = UV^*V\Sigma V^* = WP$ as required.

   (b) Show that if B is a positive-semidefinite Hermitian matrix such that $B^2 = \mu I_n$ for some nonnegative scalar μ, then $B = \sqrt{\mu}I_n$.

   - Since B is Hermitian, by the spectral theorem it is diagonalizable. If its eigenvalues are $\lambda_1, \dots, \lambda_n$ then by the spectral mapping theorem the eigenvalues of B^2 are $\lambda_1^2, \dots, \lambda_n^2$, so $\lambda_1^2 = \cdots = \lambda_n^2 = \mu$ hence each $\lambda_i = \pm\sqrt{\mu}$.
   - But since B is positive-semidefinite, all its eigenvalues are nonnegative, so in fact each $\lambda_i = \sqrt{\mu}$. This means the diagonalization of B is simply $\sqrt{\mu}I_n$: but then $B = P(\sqrt{\mu}I_n)P^{-1} = \sqrt{\mu}PP^{-1} = \sqrt{\mu}I_n$, as claimed.

   (c) Show that if A is a positive-semidefinite Hermitian matrix, then there exists a unique positive-semidefinite Hermitian matrix B satisfying $B^2 = A$ (i.e., a "square root" of A). [Hint: Reduce to the case where A is diagonal, and then use part (b) along with 5(a) from homework 8 on each eigenspace of A.]

   - Since P is positive-definite and Hermitian, by the spectral theorem we may diagonalize it via an orthonormal change of basis. So by applying this change of basis, we may assume that A is diagonal with nonnegative diagonal entries.
   - Then there clearly exists a matrix B of the desired form, namely, the diagonal matrix whose diagonal entries are the nonnegative square roots of the diagonal entries of A.
   - For uniqueness now suppose that $B^2 = A$ is diagonal. By problem 5(a) of homework 8, since $AB = BA$ we see that B maps all of the eigenspaces of A to themselves. When restricted to the μ-eigenspace of A, we see that B is still positive-semidefinite and Hermitian, and has $B^2 = \mu I_n$, so by (b), we see that B acts as multiplication by $\sqrt{\mu}$.
   - But this applies to all of the eigenspaces of A, meaning that B is simply the diagonal matrix whose diagonal entries are the nonnegative square roots of the diagonal entries of A: thus, B is unique.

   (d) Suppose P and Q are positive-semidefinite Hermitian matrices and $P^2 = Q^2$. Show that $P = Q$.

   - This follows immediately from (c): since $P^2 = Q^2$ is positive-semidefinite and Hermitian, it has a unique positive-semidefinite square root, which is both P and Q.

   (e) Show that the polar decomposition of an invertible matrix A is unique. [Hint: Show first that P is invertible and then that $WP = ZQ$ implies $P^2 = Q^2$.]

   - If A is invertible then $P = W^*A$ is also invertible. Now suppose $A = WP = ZQ$ where W, Z are unitary and P, Q are positive (semi)definite.
   - Left-multiplying by Z^* and right-multiplying by P^{-1} then yields $Z^*W = QP^{-1}$ so QP^{-1} is unitary, hence $I_n = (QP^{-1})^*(QP) = P^{-1}Q^2P^{-1}$ which yields $P^2 = Q^2$.
   - But by (d), since P and Q are positive semidefinite and $P^2 = Q^2$, we have $P = Q$ and thus also $Z = W$. Hence the polar decomposition is unique, as claimed.

Remark: The usual procedure for finding the polar form of a complex number $z = e^{i\theta}r$ is to note that $r = \sqrt{|z|^2} = \sqrt{\bar{z}z}$ and then $e^{i\theta} = z/r$. For the polar decomposition $A = WP$ we have an analogous formula: $P = \sqrt{A^*A}$, where the square root here denotes the positive-semidefinite matrix square root of (c), and when P is positive-definite we obtain the unitary part W via $W = AP^{-1}$.