1. Identify each of the following statements as true or false:

   (a) Every real Hermitian matrix is diagonalizable.
   - True: by the spectral theorem, Hermitian matrices are diagonalizable.

   (b) Every real symmetric matrix is diagonalizable.
   - True: real symmetric matrices are Hermitian, so they are diagonalizable.

   (c) Every complex Hermitian matrix is diagonalizable.
   - True: again by the spectral theorem, Hermitian matrices are diagonalizable.

   (d) Every complex symmetric matrix is diagonalizable.
   - False: $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ is not diagonalizable: its Jordan form has a 2×2 block with eigenvalue 0.

   (e) If $V = \mathbb{R}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ is the usual inner product on $\mathbb{R}^2$, then Φ is a bilinear form on V.
   - True: it is linear in both components, so it is a bilinear form.

   (f) If $V = \mathbb{C}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \overline{\mathbf{w}}$ is the usual inner product on $\mathbb{C}^2$, then Φ is a bilinear form on V.
   - False: it is not linear in the second component, so it is not a bilinear form.

   (g) If $V = \mathbb{R}$ and $\Phi(x, y) = x + 2y$, then Φ is a bilinear form on V.
   - False: although this is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$, it is not a bilinear form because it doesn't respect addition or scalar multiplication in the individual components. For example, $\Phi(1, 1) + \Phi(2, 1) \neq \Phi(3, 1)$.

   (h) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 x f(x) g(x)\, dx$, then Φ is a bilinear form on V.
   - True: it is linear in both functions, so it is a bilinear form.

   (i) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 f'(x) g'(x)\, dx$, then Φ is a bilinear form on V.
   - True: derivatives are linear, so it is still linear in both functions and thus a bilinear form.

   (j) If Φ is a symmetric bilinear form, then $[\Phi]_\beta$ is a symmetric matrix for any basis β.
   - True: the (i, j)-entry in $[\Phi]_\beta$ is equal to $\Phi(\beta_i, \beta_j)$, so if Φ is symmetric then this also equals the (j, i)-entry $\Phi(\beta_j, \beta_i)$.

   (k) If $[\Phi]_\beta$ is a symmetric matrix for some basis β, then Φ is a symmetric bilinear form.
   - True: if $[\Phi]_\beta$ is symmetric then $\Phi(\beta_i, \beta_j) = \Phi(\beta_j, \beta_i)$ for all i, j, meaning that Φ is symmetric on all basis elements. Then by linearity, Φ is symmetric on all pairs of elements in V.

   (l) If $\mathcal{B}(V)$ is the space of all bilinear forms on V and $\dim_F(V) = n$, then $\dim_F \mathcal{B}(V) = 2n$.
   - False: by taking the associated matrix with respect to a fixed basis β we showed that $\mathcal{B}(V)$ is isomorphic to the space of $n \times n$ matrices, which has dimension n^2, not $2n$.

   (m) Congruent matrices have the same eigenvalues.
   - False: for example, I_n and $4I_n$ are congruent (take $Q = 2I_n$), but their eigenvalues are different.

   (n) Congruent matrices have the same eigenvectors.
   - False: congruence also does not preserve eigenvectors.

   (o) Every $n \times n$ symmetric matrix over $\mathbb{R}$ is congruent to a diagonal matrix.
   - True: this follows from the spectral theorem or our characterization of diagonalizable bilinear forms.

   (p) Every $n \times n$ symmetric matrix over an arbitrary field F is congruent to a diagonal matrix.
   - False: if $\text{char}(F) = 2$ then this result can be false: for example, we showed $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a symmetric matrix that is not congruent to a diagonal matrix over $\mathbb{F}_2$.

2. Solve each system of differential equations:

(a) Find the general solution to $y_1' = 7y_1 + y_2$ and $y_2' = 9y_1 - y_2$.

- The coefficient matrix is $A = \begin{bmatrix} 7 & 1 \\ 9 & -1 \end{bmatrix}$ with characteristic polynomial is $p(t) = \det(tI - A) = (t-8)(t+2)$, so the eigenvalues are $\lambda = -2, 8$.
- For $\lambda = 8$, the eigenspace is 1-dimensional and spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- For $\lambda = -2$ the eigenspace is also 1-dimensional and spanned by $\begin{bmatrix} -1 \\ 9 \end{bmatrix}$.
- By the eigenvalue method, the general solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8x} + C_2 \begin{bmatrix} -1 \\ 9 \end{bmatrix} e^{-2x}}$.

(b) Find the general solution to $y_1' = 3y_1 - 2y_2$ and $y_2' = y_1 + y_2$.

- The coefficient matrix is $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ with characteristic polynomial is $p(t) = \det(tI - A) = t^2 - 4t + 5$. By the quadratic formula, the eigenvalues are $\lambda = 2 \pm i$.
- For $\lambda = 2 + i$, the eigenspace is 1-dimensional and spanned by $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.
- For $\lambda = 2 - i$ we can take the complex conjugate to get the eigenvector $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.
- The general solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2+i)x} + C_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(2-i)x}}$.
- With real functions we get $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 e^{2x} \begin{bmatrix} \cos(x) - \sin(x) \\ \cos(x) \end{bmatrix} + C_2 e^{2x} \begin{bmatrix} \sin(x) + \cos(x) \\ \sin(x) \end{bmatrix}}$.

(c) Find the general solution to $y'' - 4y = 0$. [Hint: Set $z = y'$ and convert to a system of linear equations.]

- Following the hint, if we let $z = y'$ then $z' = y'' = 4y$, so we obtain the system $\left\{ \begin{array}{rcl} y' & = & z \\ z' & = & 4y \end{array} \right\}$ with associated matrix $A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = t^2 - 4$ with roots $\lambda = -2, 2$.
- The -2 and 2-eigenspaces are spanned by $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively.
- Thus by the eigenvalue method, the general solution is $\begin{bmatrix} y \\ z \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2x} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2x}$.
- In particular, we see $y = \boxed{C_1 e^{-2x} + C_2 e^{2x}}$.

(d) Find the general solution to $y_1' = 2y_2 + \sec(2x)$ and $y_2' = -2y_1$.

- The coefficient matrix for the homogeneous system is $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ with eigenvalues $\lambda = \pm 2i$. Row-reducing to find eigenvectors yields the complex-valued solution basis $\begin{bmatrix} -i \\ 1 \end{bmatrix} e^{2ix}, \begin{bmatrix} i \\ 1 \end{bmatrix} e^{-2ix}$ with equivalent real-valued solution basis $\begin{bmatrix} \sin(2x) \\ \cos(2x) \end{bmatrix}, \begin{bmatrix} -\cos(2x) \\ \sin(2x) \end{bmatrix}$.
- We want $\tilde{\mathbf{y}} = c_1(x) \begin{bmatrix} \sin(2x) \\ \cos(2x) \end{bmatrix} + c_2(x) \begin{bmatrix} -\cos(2x) \\ \sin(2x) \end{bmatrix}$ for $\begin{bmatrix} \sin(2x) & -\cos(2x) \\ \cos(2x) & \sin(2x) \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \sec(2x) \end{bmatrix}$.
- Left-multiply by $\begin{bmatrix} \sin(2x) & -\cos(2x) \\ \cos(2x) & \sin(2x) \end{bmatrix}$, so $\begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} \sin(2x) & -\cos(2x) \\ \cos(2x) & \sin(2x) \end{bmatrix} \begin{bmatrix} 0 \\ \sec(2x) \end{bmatrix} = \begin{bmatrix} -1 \\ \tan(2x) \end{bmatrix}$ and now taking antiderivatives yields $c_1(x) = C_1 - x$ and $c_2(x) = C_2 + \frac{1}{2}\ln(\sec(2x))$.
- The solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{(C_2 + \frac{1}{2}\ln(\sec 2x)) \begin{bmatrix} \sin(2x) \\ \cos(2x) \end{bmatrix} + (C_1 - x) \begin{bmatrix} -\cos(2x) \\ \sin(2x) \end{bmatrix}}$.

(e) Solve the system $\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}$, where $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

- The coefficient matrix is $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, which is already in Jordan canonical form. Using the matrix exponential formula, we compute $e^{Ax} = \begin{bmatrix} e^{2x} & xe^{2x} & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & e^{3x} \end{bmatrix}$.
- Then desired solution is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = e^{Ax}\mathbf{y}(0) = \begin{bmatrix} e^{2x} & xe^{2x} & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & e^{3x} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2e^{2x} + 3xe^{2x} \\ 3e^{2x} \\ -e^{3x} \end{bmatrix}}$.

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3. For each bilinear form on each given vector space, compute $[\Phi]_\beta$ for the given basis β:

(a) $\Phi((a, b, c), (d, e, f)) = ad + ae - 2be + 3cd + cf$ on $V = F^3$ with $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

- We evaluate $\Phi(\beta_i, \beta_j)$ to compute the (i, j)-entry, where $\beta_1 = (1, 0, 0)$, $\beta_2 = (0, 1, 0)$, $\beta_3 = (0, 0, 1)$.
- For example, $\Phi(\beta_1, \beta_2) = \Phi((1, 0, 0), (0, 1, 0)) = 1$, while $\Phi(\beta_2, \beta_2) = \Phi((0, 1, 0), (0, 1, 0)) = -2$.
- The end result is $[\Phi]_\beta = \boxed{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}}$

(b) $\Phi(p, q) = p(-1)q(2)$ on $V = P_3(\mathbb{R})$ with $\beta = \{1, x, x^2, x^3\}$.

- We simply evaluate $\Phi(\beta_i, \beta_j)$ to compute the (i, j)-entry, where $\beta_1 = 1$, $\beta_2 = x$, $\beta_3 = x^2$, $\beta_4 = x^3$.
- For example, $\Phi(\beta_1, \beta_2) = 1 \cdot 2 = 2$, while $\Phi(\beta_2, \beta_2) = (-1) \cdot 2 = -2$ and $\Phi(\beta_4, \beta_3) = (-1)^3 2^2 = -4$.
- The end result is $[\Phi]_\beta = \boxed{\begin{bmatrix} 1 & 2 & 4 & 8 \\ -1 & -2 & -4 & -8 \\ 1 & 2 & 4 & 8 \\ -1 & -2 & -4 & -8 \end{bmatrix}}$.

(c) $\Phi(A, B) = \operatorname{tr}(AB)$ on $V = M_{2\times 2}(\mathbb{C})$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

- As above we just compute the 16 possible values $\Phi(\beta_i, \beta_j)$ for basis elements β_i, β_i.
- For example, $\Phi(\beta_1, \beta_2) = \operatorname{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = 0$, $\Phi(\beta_2, \beta_3) = \operatorname{tr}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = 1$.
- The end result is $[\Phi]_\beta = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$.

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4. For each symmetric matrix S, find an invertible rational matrix Q and diagonal matrix D such that $Q^T SQ = D$ (for emphasis, the entries in D and Q must be rational numbers!):

(a) $S = \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix}$.

- We set up the double matrix and perform row/column operations as listed:

$$\left[\begin{array}{cc|cc} 1 & 9 & 1 & 0 \\ 9 & 7 & 0 & 1 \end{array}\right] \xrightarrow[C_2 - 9C_1]{R_2 - 9R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -74 & -9 & 1 \end{array}\right]$$

- Thus, we may take $D = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & -74 \end{bmatrix}}$ with $Q^T = \begin{bmatrix} 1 & 0 \\ -9 & 1 \end{bmatrix}$ and thus $Q = \boxed{\begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix}}$.

(b) $S = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 6 \\ -2 & 6 & 7 \end{bmatrix}$.

- We set up the double matrix and perform row/column operations as listed:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 1 & 0 \\ -2 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_2-C_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 8 & -1 & 1 & 0 \\ -2 & 8 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[C_3+2C_1]{R_3+2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 8 & -1 & 1 & 0 \\ 0 & 8 & 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow[C_3-4C_2]{R_3-4R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -29 & 6 & -4 & 1 \end{array}\right]$$

- Thus, we may take $D = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -29 \end{bmatrix}}$ with $Q^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 6 & -4 & 1 \end{bmatrix}$ and thus $Q = \boxed{\begin{bmatrix} 1 & -1 & 6 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}}$.

(c) $S = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$.

- We set up the double matrix and perform row/column operations as listed:

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_1\leftrightarrow C_2]{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 5 & 2 & 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[5\cdot C_2, 5\cdot C_3]{5\cdot R_2, 5\cdot R_3} \left[\begin{array}{ccc|ccc} 5 & 10 & 10 & 0 & 1 & 0 \\ 10 & 0 & 0 & 5 & 0 & 0 \\ 10 & 0 & 125 & 0 & 0 & 5 \end{array}\right]$$

$$\xrightarrow[C_2-2C_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 5 & 0 & 10 & 0 & 1 & 0 \\ 0 & -20 & -20 & 5 & -2 & 0 \\ 10 & -20 & 125 & 0 & 0 & 5 \end{array}\right] \xrightarrow[C_3-2C_1]{R_3-2R_1} \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & -20 & -20 & 5 & -2 & 0 \\ 0 & -20 & 105 & 0 & -2 & 5 \end{array}\right] \xrightarrow[C_3-C_2]{R_3-R_2} \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & -20 & 0 & 5 & -2 & 0 \\ 0 & 0 & 125 & -5 & 0 & 5 \end{array}\right]$$

- Thus, we may take $D = \boxed{\begin{bmatrix} 5 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 125 \end{bmatrix}}$ with $Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 5 & -2 & 0 \\ -5 & 0 & 5 \end{bmatrix}$ and thus $Q = \boxed{\begin{bmatrix} 0 & 5 & -5 \\ 1 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$.

(d) $S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

- We set up the double matrix and perform row/column operations as listed:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 3 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_2-C_1]{R_2-R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 3 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_3-C_1]{R_3-R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[C_4-C_1]{R_4-R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_3-C_2]{R_3-R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[C_4-C_2]{R_4-R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow[C_4-C_3]{R_4-R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

- Thus $D = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$ with $Q^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ so $Q = \boxed{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$.

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5. Suppose V is finite-dimensional with scalar field F and $T : V \to V$ is linear. We say the polynomial $q(x) \in F[x]$ annihilates T if $q(T) = 0$.

(a) Show that the set of polynomials in $F[x]$ annihilating T is a subspace of $F[x]$.

- Simply check the subspace criterion:
- [S1] Clearly the zero polynomial annihilates T for any T.
- [S2] If $p(x)$, $q(x)$ annihilate T, then $(p+q)(T) = p(T)+q(T) = 0+0 = 0$ so $p+q$ also annihilates T.
- [S3] If $p(x)$ annihilates T, then $(\alpha p)(T) = \alpha p(T) = \alpha 0 = 0$ so αp also annihilates T.

We define the minimal polynomial of T to be the monic polynomial $m(t) \in F[t]$ of smallest positive degree annihilating T. For example, the minimal polynomial of the identity transformation is $m(t) = t - 1$.

(b) Show that every polynomial that annihilates T is divisible by the minimal polynomial. [Hint: Use polynomial division.]

- Suppose $a(t)$ annihilates T and write $a(t) = q(t)m(t) + r(t)$ with $\deg r < \deg m$.
- Then $r(T) = a(T) - q(T)m(T) = 0$ since $a(T) = m(T) = 0$.
- Since $\deg r < \deg m$ and m has minimal positive degree, we must have $r = 0$.

(c) Conclude that the minimal polynomial divides the characteristic polynomial.

- The Cayley-Hamilton theorem says that the characteristic polynomial annihilates T, so by (b), it is divisible by the minimal polynomial.

(d) Suppose λ is an eigenvalue of T. Prove that λ is a root of the minimal polynomial of T, and deduce that the minimal polynomial and the characteristic polynomial have the same set of roots. [Hint: Consider the Jordan form of an associated matrix A.]

- Consider the Jordan form of any associated matrix A to T. Since λ is an eigenvalue of A, it appears on the diagonal of the Jordan form. Then the corresponding diagonal entry of $m(J)$ will be $m(\lambda)$.
- But since $m(J) = m(PAP^{-1}) = p \cdot m(A) \cdot P^{-1} = 0$, this means $m(\lambda) = 0$ so λ is a root of m.
- Since the roots of the characteristic polynomial are the eigenvalues, all its roots are roots of m, and since m divides p, all its roots are roots of p. Thus m and p have the same roots.

(e) Parts (c) and (d) give a method to find the minimal polynomial, namely, test divisors of the characteristic polynomial with the same roots (though possibly with lower multiplicities). Find the minimal polynomials of $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$.

- $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$: The characteristic polynomial is $(x-1)^2$ and $x-1$ does not annihilate this matrix, so the minimal polynomial must be $(x-1)^2$.
- $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$: The characteristic polynomial is $(x-1)(x-2)^2$ and $(x-1)(x-2)$ does not annihilate this matrix, so the minimal polynomial must be $(x-1)(x-2)^2$.
- $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$: The characteristic polynomial is $(x-1)(x-2)^2$ and $(x-1)(x-2)$ annihilates this matrix, so the minimal polynomial must be $(x-1)(x-2)$.

(f) Show that similar matrices have the same minimal polynomial.

- If $p(A) = 0$ then conjugating yields $p(PAP^{-1}) = 0$, and conversely if $p(PAP^{-1}) = 0$ then conjugating by P^{-1} yields $p(A) = 0$. Thus the polynomials annihilating A and PAP^{-1} are the same, and so the minimal polynomials are also the same.

(g) Show that the minimal polynomial of the $k \times k$ Jordan block with eigenvalue λ is $m(t) = (t-\lambda)^k$.

- Note $J - \lambda I_k$ is the matrix N with 1s directly above the diagonal. It follows by an easy induction that N^{k-1} is not zero (it is the matrix with a single 1 in the upper right corner) but N^k is.
- Thus, $(t-\lambda)^{k-1}$ does not annihilate J, so by (c) and (d) the minimal polynomial must be $(t-\lambda)^k$.

(h) Show that the exponent of $t-\lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ in the Jordan canonical form of A.

- By (f), the minimal polynomial of a $k \times k$ Jordan block is $(t-\lambda)^k$.
- By (g), the minimal polynomial of A is the same as the minimal polynomial of its Jordan form.
- Now we simply observe that for a block-diagonal matrix, the minimal polynomial of the full matrix is the least common multiple of the minimal polynomial of each block on the diagonal, since each block must individually be annihilated by the minimal polynomial.
- Putting all of this together shows immediately that the exponent of $t-\lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ.

(i) Show that a matrix is diagonalizable over $\mathbb{C}$ if and only if its minimal polynomial has no repeated roots.

- A matrix is diagonalizable if and only if all of the blocks in its Jordan form have size 1.
- But by (h), the exponent of $t-\lambda$ in the minimal polynomial is the size of the largest Jordan block.
- Thus, A is diagonalizable if and only if the exponent of $t-\lambda$ is 1, for every eigenvalue λ. Since all eigenvalues are roots of the minimal polynomial by (d), the result follows immediately.

(j) Show that the minimal polynomial of a 2×2 matrix uniquely determines its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 2×2 matrices with minimal polynomials $m(t) = t^2 + t$, $t^2 + 1$, and $t - 3$ over $\mathbb{C}$.

- If $m = (t-\lambda)^2$ then there is a 2×2 Jordan λ-block. If $m = (t-\lambda)(t-\mu)$ then there is a 1×1 block for both λ and μ. And if $m = t - \lambda$ then there are only 1×1 λ-blocks. In each case the Jordan form is determined completely.
- The Jordan forms are $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, and $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

(k) Show the minimal and characteristic polynomials of a 3×3 matrix together uniquely determine its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 3×3 matrices with $(m(t), p(t))$ equal to (t, t^3), (t^2, t^3), (t^3, t^3), $(t^2 - t, t^3 - t^2)$, $(t^2 - t, t^3 - 2t^2 + t)$.

- If the minimal and characteristic polynomials are given, then all of the eigenvalues are known and the smallest and largest Jordan block sizes are also known. Since the only possible lists of sizes are $\{3\}$, $\{2, 1\}$, $\{2\}$, $\{1, 1, 1\}$, $\{1, 1\}$, $\{1\}$, knowing the total number of eigenvalues and the smallest and largest block sizes uniquely determines which sizes appear for each eigenvalue.
- The Jordan forms are $\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$.

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6. For $A, B \in M_{n\times n}(F)$, recall that we say A is congruent to B when there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^T A Q$. Prove that congruence is an equivalence relation on $M_{n\times n}(F)$.

- Reflexive: For $Q = I_n$ we have $A = Q^T A Q$, so A is congruent to itself.
- Symmetric: If A is congruent to B, with $B = Q^T A Q$ then since Q is invertible we can write $A = (Q^T)^{-1} B Q^{-1} = R^T B R$ where $R = Q^{-1}$, so B is congruent to A.
- Transitive: If A is congruent to B with $B = Q^T A Q$ and B is congruent to C with $C = R^T B R$, then we have $C = R^T Q^T A Q R = (QR)^T A (QR)$, and so A is congruent to C.

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7. Suppose $T : V \to V$ is a linear operator on the real inner product space V with inner product $\langle \cdot, \cdot \rangle$. Define the map $\Phi : V \times V \to F$ by setting $\Phi(\mathbf{v}, \mathbf{w}) = \langle T(\mathbf{v}), \mathbf{w} \rangle$.

(a) Show that Φ is a bilinear form on V.

- We see that $\Phi(\mathbf{v}_1 + \alpha\mathbf{v}_2, \mathbf{w}) = \langle T(\mathbf{v}_1 + \alpha\mathbf{v}_2), \mathbf{w} \rangle = \langle T(\mathbf{v}_1) + \alpha T(\mathbf{v}_2), \mathbf{w} \rangle = \langle T(\mathbf{v}_1), \mathbf{w} \rangle + \alpha \langle T(\mathbf{v}_2), \mathbf{w} \rangle = \Phi(\mathbf{v}_1, \mathbf{w}) + \alpha\Phi(\mathbf{v}_2, \mathbf{w})$, so Φ is linear in the first coordinate.
- Likewise, $\Phi(\mathbf{v}, \mathbf{w}_1 + \alpha\mathbf{w}_2) = \langle T(\mathbf{v}), \mathbf{w}_1 + \alpha\mathbf{w}_2 \rangle = \langle T(\mathbf{v}), \mathbf{w}_1 \rangle + \alpha \langle T(\mathbf{v}), \mathbf{w}_2 \rangle = \Phi(\mathbf{v}, \mathbf{w}_1) + \alpha\Phi(\mathbf{v}_2, \mathbf{w}_2)$, so Φ is linear in the second coordinate.

(b) Show that Φ is symmetric if and only if T is Hermitian.

- Observe that, since the inner product is real (and therefore symmetric), we have $\Phi(\mathbf{w},\mathbf{v})-\Phi(\mathbf{v},\mathbf{w}) = \langle T(\mathbf{w}),\mathbf{v}\rangle - \langle T(\mathbf{v}),\mathbf{w}\rangle = \langle \mathbf{w}, T^*(\mathbf{v})\rangle - \langle \mathbf{w}, T(\mathbf{v})\rangle = \langle \mathbf{w}, (T^*-T)(\mathbf{v})\rangle$.
- Thus, if $T^* = T$ then we see immediately that $\Phi(\mathbf{w},\mathbf{v}) = \Phi(\mathbf{v},\mathbf{w})$ so Φ is symmetric.
- Conversely, if Φ is symmetric, then $\langle \mathbf{w}, (T^*-T)(\mathbf{v})\rangle = 0$ for all $\mathbf{v}, \mathbf{w}$, so by one of our lemmas, this means $T^* - T = 0$ and thus $T^* = T$.

(c) If V is finite-dimensional, prove that Φ is an inner product on V if and only if T is Hermitian and all its eigenvalues are positive. [Hint: Use [I3] and the spectral theorem.]

- We know that Φ is linear, which is condition [I1]. In order to satisfy [I2], it must also be symmetric, and that is equivalent to saying that T is Hermitian by part (b).
- For [I3], we also require $\Phi(\mathbf{v},\mathbf{v}) \geq 0$ for all $\mathbf{v}$, which requires $\langle T(\mathbf{v}),\mathbf{v}\rangle \geq 0$ for all $\mathbf{v} \in V$ with equality if and only if $\mathbf{v} = \mathbf{0}$. Since T is diagonalizable, in particular this holds for every eigenvector $\mathbf{v}$ of T: but if $T(\mathbf{v}) = \lambda\mathbf{v}$ then we have $\lambda = \langle T(\mathbf{v}),\mathbf{v}\rangle / \langle \mathbf{v},\mathbf{v}\rangle > 0$, and so $\lambda > 0$.
- Thus, the positive-eigenvalue condition is necessary. On the other hand, it is also sufficient: since T is Hermitian, it is diagonalizable on V by the spectral theorem, and so we have a orthonormal basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T.
- If $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ then we can explicitly compute $\langle T(\mathbf{v}),\mathbf{v}\rangle = a_1^2\lambda_1 + \cdots + a_n^2\lambda_n$, which is indeed positive when $\mathbf{v} \neq \mathbf{0}$. Thus, the given conditions are both necessary and sufficient.

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8. [Challenge] Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The goal of this problem is to prove the Courant-Fischer theorem: that $\lambda_i = \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v})$ for each $1 \leq i \leq n$. This characterization of the eigenvalues in terms of a min-max property is useful in practical computations, particularly the $i = 1$ case: $\lambda_1 = \max_{||\mathbf{v}||=1}(\mathbf{v}^* A\mathbf{v})$.

(a) Show that it suffices to prove the Courant-Fischer theorem when the matrix A is diagonal.

- As we showed, Hermitian matrices are orthogonally diagonalizable.
- Applying the corresponding orthonormal change of basis does not change the given min-max condition, since it is independent of basis, so the general result holds if the diagonal result holds.

Per (a), we now assume that A is diagonal and that for $\mathbf{v} = (x_1, \dots, x_n)$ we have $\mathbf{v}^* A\mathbf{v} = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$.

(b) Show that $\lambda_i \geq \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v})$. [Hint: Take W to be the subspace spanned by the first $i-1$ coordinate vectors.]

- Per the hint suppose W is the subspace spanned by the first $i-1$ coordinate vectors.
- Then any $\mathbf{v} \in W^\perp$ has its first $i-1$ coordinates equal to zero, and thus $\mathbf{v}^* A\mathbf{v} = \lambda_i x_i^2 + \cdots + \lambda_n x_n^2 \leq \lambda_i(x_i^2 + \cdots + x_n^2) \leq \lambda_i \,||\mathbf{v}||^2 = \lambda_i$.
- Equality can occur when $\mathbf{v}$ is equal to the ith unit coordinate vector, so we see $\max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v}) = \lambda_i$ for this choice of W.
- Therefore, we have $\lambda_i \geq \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v})$ since the minimum includes this specific W where the value of the max-term is λ_i.

(c) Prove that $\lambda_i \leq \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v})$. [Hint: For any W of dimension $i-1$, let V_i be the subspace spanned by the first i coordinate vectors and take $\mathbf{v} \in V_i \cap W^\perp$.]

- Per the hint let V_i be the subspace spanned by the first i coordinate vectors, and let W be any $(i-1)$-dimensional subspace. Then $\dim(W^\perp) = n - (i-1) = n + 1 - i$, so $W_i \cap W^\perp$ is nonempty since the dimensions sum to more than $\dim(V) = n$.
- Now let $\mathbf{v}$ be a unit vector in $V_i \cap W^\perp$. Then $\mathbf{v}^* A\mathbf{v} = \lambda_1 x_1^2 + \cdots + \lambda_i x_i^2 \geq \lambda_i(x_1^2 + \cdots + x_i^2) = \lambda_i$, meaning that $\lambda_i \leq \max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v})$.
- Since this holds for arbitrary W we see $\lambda_i \leq \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v})$.

(d) Deduce that $\lambda_i = \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v}\in W^\perp}(\mathbf{v}^* A \mathbf{v})$ for each i.

- This follows immediately from (a), (b), and (c).
