E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 10, due Fri Apr 10th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:
   (a) Every real Hermitian matrix is diagonalizable.
   (b) Every real symmetric matrix is diagonalizable.
   (c) Every complex Hermitian matrix is diagonalizable.
   (d) Every complex symmetric matrix is diagonalizable.
   (e) If $V = \mathbb{R}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ is the usual inner product on $\mathbb{R}^2$, then Φ is a bilinear form on V.
   (f) If $V = \mathbb{C}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \overline{\mathbf{w}}$ is the usual inner product on $\mathbb{C}^2$, then Φ is a bilinear form on V.
   (g) If $V = \mathbb{R}$ and $\Phi(x, y) = x + 2y$, then Φ is a bilinear form on V.
   (h) If $V = C[0,1]$ and $\Phi(f, g) = \int_0^1 x f(x) g(x)\, dx$, then Φ is a bilinear form on V.
   (i) If $V = C[0,1]$ and $\Phi(f, g) = \int_0^1 f'(x) g'(x)\, dx$, then Φ is a bilinear form on V.
   (j) If Φ is a symmetric bilinear form, then $[\Phi]_\beta$ is a symmetric matrix for any basis β.
   (k) If $[\Phi]_\beta$ is a symmetric matrix for some basis β, then Φ is a symmetric bilinear form.
   (l) If $\mathcal{B}(V)$ is the space of all bilinear forms on V and $\dim_F(V) = n$, then $\dim_F \mathcal{B}(V) = 2n$.
   (m) Congruent matrices have the same eigenvalues.
   (n) Congruent matrices have the same eigenvectors.
   (o) Every $n \times n$ symmetric matrix over $\mathbb{R}$ is congruent to a diagonal matrix.
   (p) Every $n \times n$ symmetric matrix over an arbitrary field F is congruent to a diagonal matrix.

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2. Solve each system of differential equations:
   (a) Find the general solution to $y_1' = 7y_1 + y_2$ and $y_2' = 9y_1 - y_2$.
   (b) Find the general solution to $y_1' = 3y_1 - 2y_2$ and $y_2' = y_1 + y_2$.
   (c) Find the general solution to $y'' - 4y = 0$. [Hint: Set $z = y'$ and convert to a system of linear equations.]
   (d) Find the general solution to $y_1' = 2y_2 + \sec(2x)$ and $y_2' = -2y_1$.
   (e) Solve the system $\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}$, where $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

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3. For each bilinear form on each given vector space, compute $[\Phi]_\beta$ for the given basis β:
   (a) $\Phi((a,b,c),(d,e,f)) = ad + ae - 2be + 3cd + cf$ on $V = F^3$ with $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$.
   (b) $\Phi(p, q) = p(-1)q(2)$ on $V = P_3(\mathbb{R})$ with $\beta = \{1, x, x^2, x^3\}$.
   (c) $\Phi(A, B) = \mathrm{tr}(AB)$ on $V = M_{2\times 2}(\mathbb{C})$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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4. For each symmetric matrix S, find an invertible rational matrix Q and diagonal matrix D such that $Q^TSQ = D$ (for emphasis, the entries in D and Q must be rational numbers!):

(a) $S = \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix}$. (b) $S = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 6 \\ -2 & 6 & 7 \end{bmatrix}$. (c) $S = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$. (d) $S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose V is finite-dimensional with scalar field F and $T : V \to V$ is linear. We say the polynomial $q(x) \in F[x]$ annihilates T if $q(T) = 0$.

   (a) Show that the set of polynomials in $F[x]$ annihilating T is a subspace of $F[x]$.

   We define the minimal polynomial of T to be the monic polynomial $m(t) \in F[t]$ of smallest positive degree annihilating T. For example, the minimal polynomial of the identity transformation is $m(t) = t - 1$.

   (b) Show that every polynomial that annihilates T is divisible by the minimal polynomial. [Hint: Use polynomial division.]
   (c) Conclude that the minimal polynomial divides the characteristic polynomial.
   (d) Suppose λ is an eigenvalue of T. Prove that λ is a root of the minimal polynomial of T, and deduce that the minimal polynomial and the characteristic polynomial have the same set of roots. [Hint: Consider the Jordan form of an associated matrix A.]
   (e) Parts (c) and (d) give a method to find the minimal polynomial, namely, test divisors of the characteristic polynomial with the same roots (though possibly with lower multiplicities). Find the minimal polynomials of $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$.
   (f) Show that similar matrices have the same minimal polynomial.
   (g) Show that the minimal polynomial of the $k \times k$ Jordan block with eigenvalue λ is $m(t) = (t - \lambda)^k$.
   (h) Show that the exponent of $t - \lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ in the Jordan canonical form of A.
   (i) Show that a matrix is diagonalizable over $\mathbb{C}$ if and only if its minimal polynomial has no repeated roots.
   (j) Show that the minimal polynomial of a 2×2 matrix uniquely determines its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 2×2 matrices with minimal polynomials $m(t) = t^2 + t$, $t^2 + 1$, and $t - 3$ over $\mathbb{C}$.
   (k) Show the minimal and characteristic polynomials of a 3×3 matrix together uniquely determine its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 3×3 matrices with $(m(t), p(t))$ equal to (t, t^3), (t^2, t^3), (t^3, t^3), $(t^2 - t, t^3 - t^2)$, $(t^2 - t, t^3 - 2t^2 + t)$.

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6. For $A, B \in M_{n\times n}(F)$, recall that we say A is congruent to B when there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^TAQ$. Prove that congruence is an equivalence relation on $M_{n\times n}(F)$.

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7. Suppose $T : V \to V$ is a linear operator on the real inner product space V with inner product $\langle \cdot, \cdot \rangle$. Define the map $\Phi : V \times V \to F$ by setting $\Phi(\mathbf{v}, \mathbf{w}) = \langle T(\mathbf{v}), \mathbf{w} \rangle$.

   (a) Show that Φ is a bilinear form on V.
   (b) Show that Φ is symmetric if and only if T is Hermitian.
   (c) If V is finite-dimensional, prove that Φ is an inner product on V if and only if T is Hermitian and all its eigenvalues are positive. [Hint: Use [I3] and the spectral theorem.]

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8. [Challenge] Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The goal of this problem is to prove the Courant-Fischer theorem: that $\lambda_i = \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v} \in W^\perp} (\mathbf{v}^* A \mathbf{v})$ for each $1 \leq i \leq n$. This characterization of the eigenvalues in terms of a min-max property is useful in practical computations, particularly the $i = 1$ case: $\lambda_1 = \max_{||\mathbf{v}||=1} (\mathbf{v}^* A \mathbf{v})$.

   (a) Show that it suffices to prove the Courant-Fischer theorem when the matrix A is diagonal.

   Per (a), we now assume that A is diagonal and that for $\mathbf{v} = (x_1, \dots, x_n)$ we have $\mathbf{v}^* A \mathbf{v} = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$.

   (b) Show that $\lambda_i \geq \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v} \in W^\perp} (\mathbf{v}^* A \mathbf{v})$. [Hint: Take W to be the subspace spanned by the first $i - 1$ coordinate vectors.]

   (c) Prove that $\lambda_i \leq \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v} \in W^\perp} (\mathbf{v}^* A \mathbf{v})$. [Hint: For any W of dimension $i - 1$, let V_i be the subspace spanned by the first i coordinate vectors and take $\mathbf{v} \in V_i \cap W^\perp$.]

   (d) Deduce that $\lambda_i = \min_{\dim W = i-1} \max_{||\mathbf{v}||=1, \mathbf{v} \in W^\perp} (\mathbf{v}^* A \mathbf{v})$ for each i.
