E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Homework 1, due Tue Jan 13th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:

   (a) The set of integers $\mathbb{Z}$ is not a field.
   (b) Every field has infinitely many elements.
   (c) It is impossible to have $6 = 0$ in a field F.
   (d) For any $n \times n$ matrices A and B, $(A+B)^2 = A^2 + 2AB + B^2$.
   (e) For any $n \times n$ matrices A and B, $(BA)^T = B^T A^T$.
   (f) For any invertible $n \times n$ matrices A and B, $(A+B)^{-1} = A^{-1} + B^{-1}$.
   (g) For any invertible $n \times n$ matrices A and B, $(BA)^{-1} = A^{-1}B^{-1}$.
   (h) If A and B are $n \times n$ matrices with $\det(A) = 2$ and $\det(B) = 3$, then $\det(AB) = 6$.
   (i) If A is an $n \times n$ matrix with $\det(A) = 3$, then $\det(2A) = 3n$.
   (j) For any $n \times n$ matrix A, $\det(A) = -\det(A^T)$.
   (k) For any $n \times n$ matrices A and B, $\det(AB) = \det(B)\det(A)$.

---

2. Find the general solution to each system of linear equations:

   (a) $\left\{ \begin{array}{r} -x - 3y + 5z = 9 \\ 3x + 2y + 2z = 0 \\ 2x + 2y + 3z = 4 \end{array} \right\}$.

   (b) $\left\{ \begin{array}{r} x - 2y + 4z = 4 \\ 2x + 4y + 8z = 0 \end{array} \right\}$.

   (c) $\left\{ \begin{array}{l} a + b + c + d \quad\;\; = 2 \\ a + b + c \quad\; + e = 3 \\ a + b \quad + d + e = 4 \\ a \quad + c + d + e = 5 \\ \quad\; b + c + d + e = 6 \end{array} \right\}$.

   (d) $\left\{ \begin{array}{r} x + 3y + \;\; z = -4 \\ -x - 6y + 8z = 10 \\ 2x + 4y + 8z = 0 \end{array} \right\}$.

   (e) $\left\{ \begin{array}{rcr} a + b + c + d + e & = & 1 \\ a + 2b + 3c + 4d + 5e & = & 6 \end{array} \right\}$.

---

3. Compute the following things:

   (a) The reduced row-echelon forms of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 & 2 & 3 \\ 2 & 1 & 0 & -1 & -2 \\ -4 & -2 & 0 & 3 & 0 \end{bmatrix}$.

   (b) The determinants of $\begin{bmatrix} -1 & 5 & 2 \\ 0 & -3 & 7 \\ 2 & 8 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \\ 8 & 27 & 64 & 125 \end{bmatrix}$.

   (c) The inverses of $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -3 & -2 \\ -3 & 7 & 8 \\ 2 & -6 & -5 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose that A and B are $n \times n$ matrices with entries from a field F.

   (a) If AB is invertible, show that A and B are invertible.

   (b) If A is invertible, show that A^T is invertible and that its inverse is $(A^{-1})^T$.

---

5. Let F be a field of characteristic not 2 (i.e., in which $2 \neq 0$). A square matrix A with entries from F is called symmetric if $A = A^T$ and skew-symmetric if $A = -A^T$.

   (a) For any $n \times n$ matrix B, show that $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric.

   (b) Show that any square matrix M can be written *uniquely* in the form $M = S + T$ where S is symmetric and T is skew-symmetric. [Make sure to prove that there is *only* one such decomposition!]

   (c) If A is a skew-symmetric $n \times n$ real matrix and n is odd, show that $\det(A) = 0$.

   (d) If A and B are symmetric, prove that AB is symmetric if and only if A and B commute (i.e., $AB = BA$).

---

6. The goal of this problem is to prove a matrix inversion formula called the Woodbury matrix identity, and then give an application.

   (a) Suppose P and Q are $n \times n$ matrices such that $I_n + QP$ is invertible. Show that $I_n + PQ$ is also invertible and that its inverse is $M = I_n - P(I_n + QP)^{-1}Q$. [Hint: Multiply out $M(I_n + PQ)$.]

   (b) Prove the Woodbury matrix identity: if A is an invertible $n \times n$ matrix, U is an $n \times k$ matrix, C is an invertible $k \times k$ matrix, and V is a $k \times n$ matrix such that $C^{-1} + VA^{-1}U$ is invertible, then $A + UCV$ is invertible and
   $$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$
   [Hint: Observe $A + UCV = A(I_n + PQ)$ where $P = A^{-1}U$ and $Q = CV$, then use (a).]

   (c) Suppose A and C are invertible $n \times n$ matrices and $A + C$ is also invertible. Show that $(A + C)^{-1} = A^{-1} - (A + AC^{-1}A)^{-1}$.

---

7. [Challenge] Let D_n denote the value of the $(n-1) \times (n-1)$ determinant $\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$.

   Determine whether $\lim_{n\to\infty} \dfrac{D_n}{n!}$ exists. [Hint: Start by subtracting the first row from the other rows.]
